Scale-Space for Union of 3D Balls

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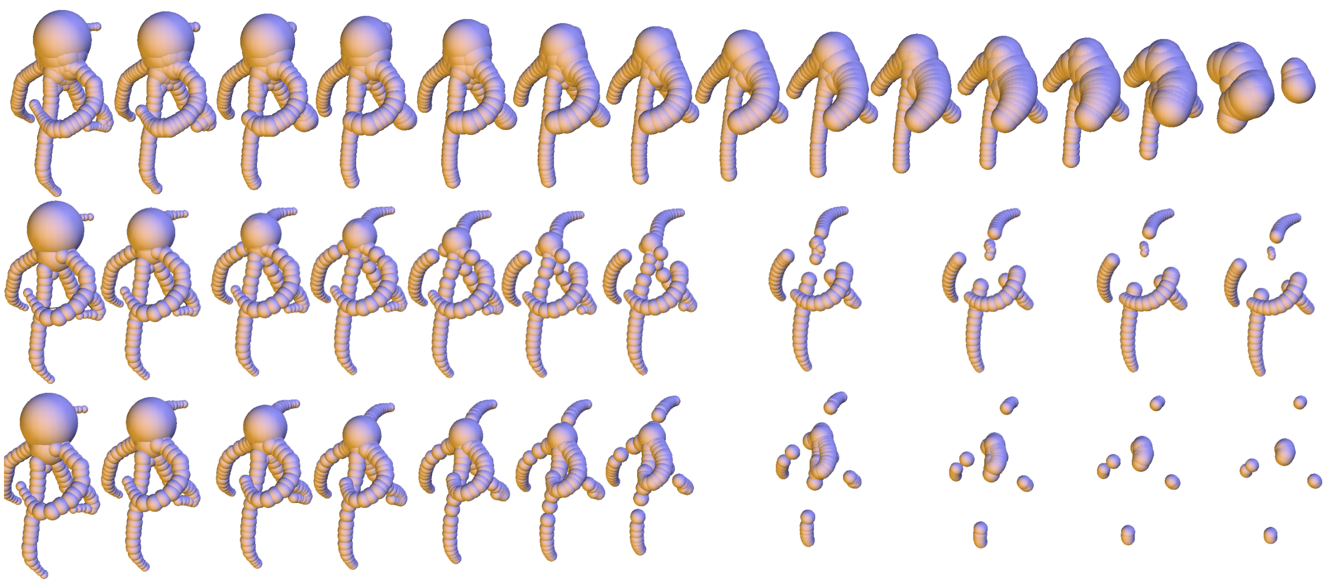


Figure 1. Iterations of smoothing of a union of balls on a meta-ball model of an octopus, at every 8 iterations: the proposed scale space (top), compared to a gaussian filtering along the medial axis (middle) and spatial gaussian filtering (bottom) : our smoothing achieves progressive detail remotion.

***Abstract*—Shape discretization through union of weighted points or balls appears as a common representation in different fields of computer graphics and geometric modeling. Among others, it has been very successful for implicit surface reconstruction with radial basis functions, molecular atomic models, fluid simulation from particle systems and deformation tracking with particle filters. These representations are commonly generated from real measurements or numerical computations, which may require filtering and smoothing operations.This work proposes a smoothing mechanism for union of balls that tries to inherit from the scale-space properties of bi-dimensional curvature motion: it avoids disconnecting the shape, prevents self-intersection, regularly decreases the area and convexifies the shape. The smoothing is computed iteratively by moving each ball of the union according to a combination of projected planar curvature motions. Experiments exhibits nice properties of this scale-space.**

***Keywords*-Union of Balls; Scale Spaces; Curvature Motion;**

I. Introduction

Although in Computer Graphics shapes are predominantly represented by triangulated surfaces, volumetric representations are spreading quickly for a wide range of applications. For example, particle-based fluid simulations offer excellent trade-offs for games and animation [26], and point sets models are easily acquired by 3d scanning processes [7], and their approximations can be implicitly defined with radial basis functions [9], [22], [14]. In particular, the shape model of union of balls is a natural representation for specific applications such as molecular simulation [24], low-density fluids [25] and shape skeletons [1], [17], [20]. In such applications however, the union of balls models commonly contain noise and redundancies. There is thus a need for tools like simplification, smoothing or multi-resolution adapted to the union of balls model. In particular, scale-spaces generated by successive smoothing are powerful tools for shape analysis.

In this work, we propose a smoothing scheme for union of balls inspired by curvature motions. The planar curvature motion smoothes curves with very desirable scale-space properties: it does not create intersection or self-intersections nor does it cut the curve, the curve becomes convex in finite time and at the end the region becomes close to a disk [4]. Those scale-space properties are not achieved in space by gaussian filters, while the proposed method exhibits better properties (see Figure 1).

Related work: To our knowledge, very few works have proposed geometric tools for generating scale-spaces of union of balls, although topological schemes based on α-shapes and persistence have been well studied [16], [13]. For the particular case of radial basis function (RBF), more concise generations have been proposed [14], placing the poles of the RBF on the Voronoï centers of the shape. Proper simplification algorithms were devised mainly for point cloud representations instead of balls [6]. In particular, curvature motion is used for point set surface simplifications [23], while we work directly on the union of balls.

The most related approaches are medial-axis based simplification. On one side, geometric approaches adaptively prunes the medial axis [27] to remove spurious branches. This process is highly non-continuous and limited to very high frequencies smoothing. On the other side, topological approaches [8] progressively reduce all the branches of the medial axis. The smoothing effect is continuous but low frequencies may disappear before high ones. Our approach is based on bi-dimensional curvature motions, which naturally generates a well-behaved scale-space, i.e. removes high frequencies in a progressive and smooth order.

Curvature motion smoothly evolves each point $\mathbf{p}$ of a curve along the normal at $\mathbf{p}$ proportionally to the curvature at that point. It is widely used for multi-resolution representations of plane regions [19], [18]. For surfaces, the mechanism is similar, although many curvatures co-exist, in particular mean, minimal and maximal curvatures. Very few properties of three-dimensional curvature motions are known, and in particular no formulation has been devised as differential equations on the medial axis. While counter-examples exist for mean curvature motion [2], experiments suggest good properties of the minimal curvature motion [10]. However in two dimensions, many properties have been stated and efficient formulations have been adapted. In particular, this work is based on the discretization of curvature motion for union of disks [15].

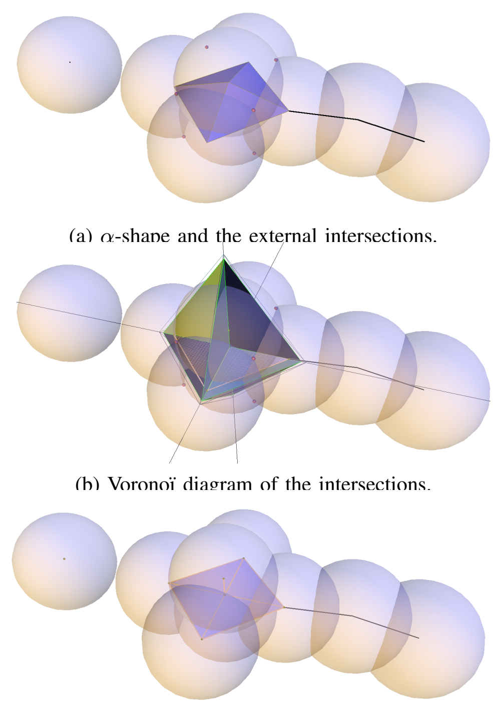
Contributions: We propose a smoothing scheme for union of balls that practically generates a well-behaved scale-space: it avoids intersections and disconnections and progressively removes details of the shape. We use a heuristic inspired by curvature motion: Since curvature motion for union of balls is well understood in the plane but not in space, we propose to approximate three-dimensional motions from their planar projections. At a given ball B, we intersect the union of balls with some equatorial planes of B and apply planar curvature motion on the intersected disks [15]. The movement of B is then computed as the mean, minimal or maximal planar motion, which should approximate three-dimensional mean, minimal or maximal curvature motion. To avoid computing the movement for all planes, we use the adjacencies of a medial ball B to define significant local planes. We further propose a simple sampling correction scheme that ensures numerical stability and avoids disconnecting the shape. This work is an extension of the M.Sc dissertation of Cynthia Ferreira [5] and Betina Vath [28], advised respectively by Marcos Craizer and Thomas Lewiner at PUC-Rio.

II. Preliminaries

In this section, we will review the definition of medial axis and how to compute it for union of balls, following the work of Amenta and Kolluri [17]. Then, we review the formulation of Teixeira [10], [11], [12] for the curvature motion on the medial axis for the bi-dimensional case. We then quickly expose how to implement these equations for union of balls, as done by Lewiner *et al.* [15].

A. Union of balls

In this work, we use the notation for union of balls of the work of Amenta and Kolluri [17]. This notation denotes identically a ball B of center $\mathbf{p}_B$ and radius r_B, as the geometric set of points $\{\mathbf{p}\ s.t.\ \|\mathbf{p}_B\mathbf{p}\| \leq r_B\}$, or as the point $\mathbf{p}_B$ weighted by r_B. This identification allows to write indifferently $\mathcal{U}$ for the geometric union of the balls, which is suitable for describing shapes and medial axis, or for the set of weighted points, which gives support for constructing α–shape or Voronoï diagrams. When the context makes it clear, we denote $\mathbf{p}_B$ by B.



(a) α-shape and the external intersections.

(b) Voronoï diagram of the intersections.

(c) Medial axis: singular part from the α-shape and regular part from the Voronoï diagram.

Figure 2. Main steps for computing the medial axis of a union of balls.

B. Medial axis

A union of balls can be concisely described in terms of its medial axis. The medial axis of a shape is the locus of the centers of balls that are tangent to the shape in two or more points, where all such balls are contained in the shape (see Figure 2). The distance r from a point M of the medial axis to the boundary of the shape is called the radius function. For smooth curves, r is a smooth function of the medial axis point. Generically, the medial axis of a discrete shape is a finite simplicial complex, i.e. a collection of vertices, open edges and triangles that do not intersect.

Following Amenta and Kolluri [17], the medial axis of a union of balls $\mathcal{U}$ can be computed from the α–shape of $\mathcal{U}$ and the Voronoï diagram of the external intersection point (see Figure 2c). The external intersections are computed from the α-shape of $\mathcal{U}$ with $\alpha = 0$. More precisely, a triangle of the α-shape links three balls that intersect transversally. The intersections are external if the triangle is singular (without adjacent tetrahedron), and only one intersection is external if the edge is on the boundary of the α-shape (see Figure 2a). Then, the medial axis of $\mathcal{U}$ is the intersection of the α-shape with the Voronoï diagram of the obtained external intersections (see Figure 2b), together with the singular part of the α-shape.

C. 2D curvature motion on the medial axis

The curvature motion of a curve is obtained by moving each point $\mathbf{p}$ of a smooth curve along its normal $\mathbf{N}(\mathbf{p})$. The magnitude of the movement is proportional to the curvature $K(\mathbf{p})$ at point $\mathbf{p}$. This can be translated into the following differential equation: $\mathbf{p}_t = K(\mathbf{p}) \cdot \mathbf{N}(\mathbf{p})$, where $\mathbf{p}_t = \frac{\partial \mathbf{p}}{\partial t}$ denotes the time derivative.

This motion deforms the curve, and thus changes the medial axis of its interior and the associated radius function r. The differential equation of this medial axis evolution has been stated by Teixeira [10]. In particular, for a regular point M of the medial axis (i.e. neither an end point nor a bifurcation), it can be written [11]:

$$\left\{ \begin{array}{l} \mathbf{M}_t = \frac{\mathcal{K}\left(1-{r_v}^2\right)}{(1-{r_v}^2-r r_{vv})^2 - r^2\mathcal{K}^2(1-{r_v}^2)}\mathcal{N} \\ r_t = \frac{r\mathcal{K}^2\left(1-{r_v}^2\right)+r_v\left(1-{r_v}^2-r r_{vv}\right)}{(1-{r_v}^2-r r_{vv})^2 - r^2\mathcal{K}^2(1-{r_v}^2)} \end{array} \right.$$

where the medial axis $\mathbf{M}(v)$ and the radius function $r(v)$ are parameterized by the arc–length v, and $\mathcal{N}$ and $\mathcal{K}$ are the normal and the curvature at the regular point M, respectively.

For an end point M of the medial axis, the equation can be written in terms of the arc length s, the derivative of the curvature K and the normal $\mathbf{N}$ of the boundary curve at the tangent point with the ball centered at M [12], [15]:

$$\left\{ \begin{array}{l} \mathbf{M}_t = -K_{ss}\mathbf{N} \\ r_t = -K_{ss} - K \end{array} \right.$$

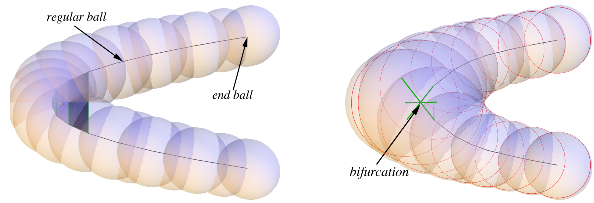


Figure 3. The 2D curvature motion for union of disk deforms bifurcation balls as means of regular motions (green lines).

D. 2D curvature motion for union of disks

The above equations can be implemented for union of balls [15], following a classification of the disks based on their adjacency in the medial axis (see Figure 3). The easy parts of union of balls is that their medial axis can be exactly computed, as described at the beginning of this section, and that the radius function r is simply the radius of the balls that are part of the medial axis. The estimates for the curvature of the medial axis, the second derivatives of the radius function naturally follow. The second derivative of the boundary curvature can be obtained by ellipse approximations at an end ball, when considering the neighboring balls in the α-shape [15]. The hard part remains numerical issues, such as noisy vs. clean end ball detection, sampling conditions and the handling of topological changes and bifurcation cases.

This last case is interesting for the present 3D extension. According to Teixeira's formulation for bifurcations [12], the movement at the intersection of three branches of the medial axis can be approximated as the weighted mean of the movement of each branch. This strategy is at the base of our approach for approximate 3D curvature motion introduced in the next section.

III. Scale-Space from Curvature Motion

We aim at smoothing union of balls, mimicking curvature motion (see Figure 4). One option would be to devise formulas of the induced motion onto the medial axis, like stated by Teixeira [10] in 2D (see Section II-C). However, such calculus would be extremely heavy due to the many singularities of 3D medial axis. We thus propose to build our smoothing filter directly on top of 2D curvature motion, which is much more studied.

Observing the nature of the equations of Section II-C, the curvature motion deforms a point of the medial axis along the normal at the medial axis. For the 3D case, we will use a similar approach, moving the medial axis in a direction perpendicular to the medial curve. Moreover, similarly to the bifurcations in 2D, we compute our scale-space by combinations of simpler curvature-based deformations. In particular at vertices of the medial axis, where only a normal cone can be defined, we smooth the surrounding shape from the motion in planes containing a normal of the cone. The different combinations of these planar motions lead to approximations of different motions, e.g. mean, minimal, maximal curvature motions.

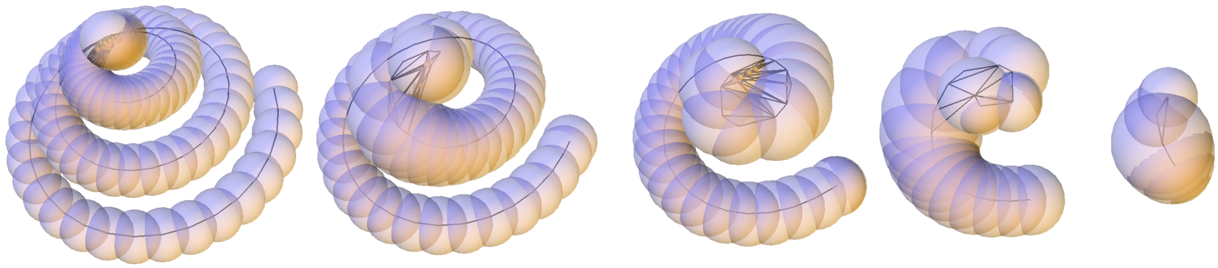


Figure 4. Our scale-space smoothing on a spirally shaped, after 1, 20, 40, 60 and 80 iterations: the movement avoids self-intersection, even in delicate cases. The lines show the medial axis edges along the defomartion.

A. Algorithm overview

This leads to the following algorithm: starting from a union of balls $\mathcal{U}$, we compute its medial axis $\mathbf{M}$, and in particular the adjacency relations of the balls inside this medial axis (see Figure 5). For each ball B of $\mathcal{U}$, we select a set of equatorial planes $\{P_i\}$ for B containing a normal of the normal cone of $\mathbf{M}$ at $\mathbf{p}_B$ (see Section III-B). For each plane P_i, we intersect $\mathcal{U}$ with P_i, generating a set of disks (see Figure 6), and we apply the 2D curvature motion on this set, as described in Section II-D. This moves the center $\mathbf{p}_B$ of B inside P_i and changes its radius, leading to a new ball B_i. We then combine these planar movements from B to B_i to obtain the new position and radius of B in 3D (see Section III-C). Similarly to the 2D case, we can optionally impose a sampling condition on the movement (see Section III-D). To avoid testing for each single equatorial plane P_i, we select sample planes as described next.

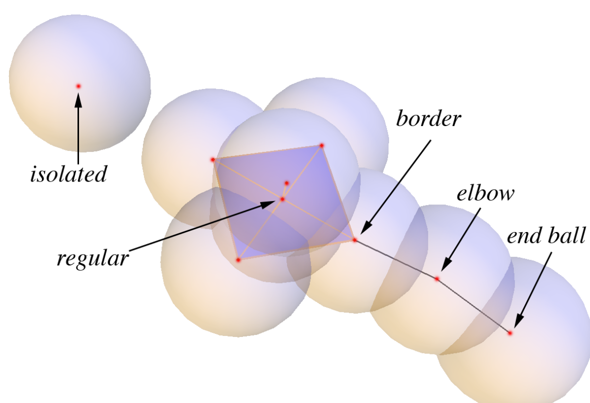


Figure 5. Classification of the balls according to their adjacency in the medial axis.

B. Ball classification and planes selection

From the adjacency relations in the medial axis $\mathbf{M}$, we can classify each ball B as follows (see Figure 5):

- *isolated* when B has no neighbor in $\mathbf{M}$. This occurs for example when a ball is in the interior of the shape, but it does not belong to the medial axis, or when the shape is reduced to that ball. In that case, the radius of B is decreased proportionally to the time step and no plane is needed.
- *end-ball* when B has only one neighbor B'. This implies that B is on the singular part of $\mathbf{M}$, either B is on a symmetric part of the shape (end of a branch) or it can be interpreted as a noisy geometry on the boundary of the shape [8]. The planes P_i must then contain the singular edge $\mathbf{p}_B\mathbf{p}_{B'}$. Since the shape is locally isotropic near $\mathbf{p}_B$, we choose planes to regularly sample all rotations, for example at 0, $45°$, $90°$ and $135°$ with respect to a fixed direction.
- *elbow* when B is on the singular part of $\mathbf{M}$ and has two adjacent neighbors B' and B'', i.e., in the middle of a branch. If $\mathbf{p}_B$, $\mathbf{p}_{B'}$ and $\mathbf{p}_{B''}$ are colinear, then we consider the same planes as the *end-ball* case. If they are not colinear, we essentially have a planar case, and we choose two planes, one passing through $\mathbf{p}_B$, $\mathbf{p}_{B'}$ and $\mathbf{p}_{B''}$ and a perpendicular one: the bisector plane of the angle $\mathbf{p}_{B'}\mathbf{p}_B\mathbf{p}_{B''}$.
- *regular* when B is on the interior of $\mathbf{M}$. In that case, all adjacent edges belong to two triangles of the medial axis. This case is similar to 2D bifurcation, and we can average the separate computation of each branch. We thus compute, for each adjacent triangle $\mathbf{p}_B\mathbf{p}_{B'}\mathbf{p}_{B''}$, two planes P_i passing through $\mathbf{p}_B$ and perpendicular to the triangle, one containing $\mathbf{p}_{B'}$ and the other containing $\mathbf{p}_{B''}$.

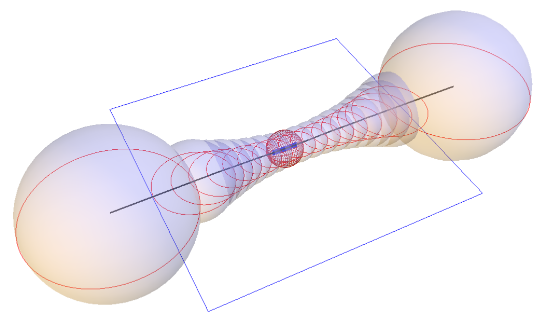


Figure 6. The motion of the central ball is a combination of its motion in planar cuts.

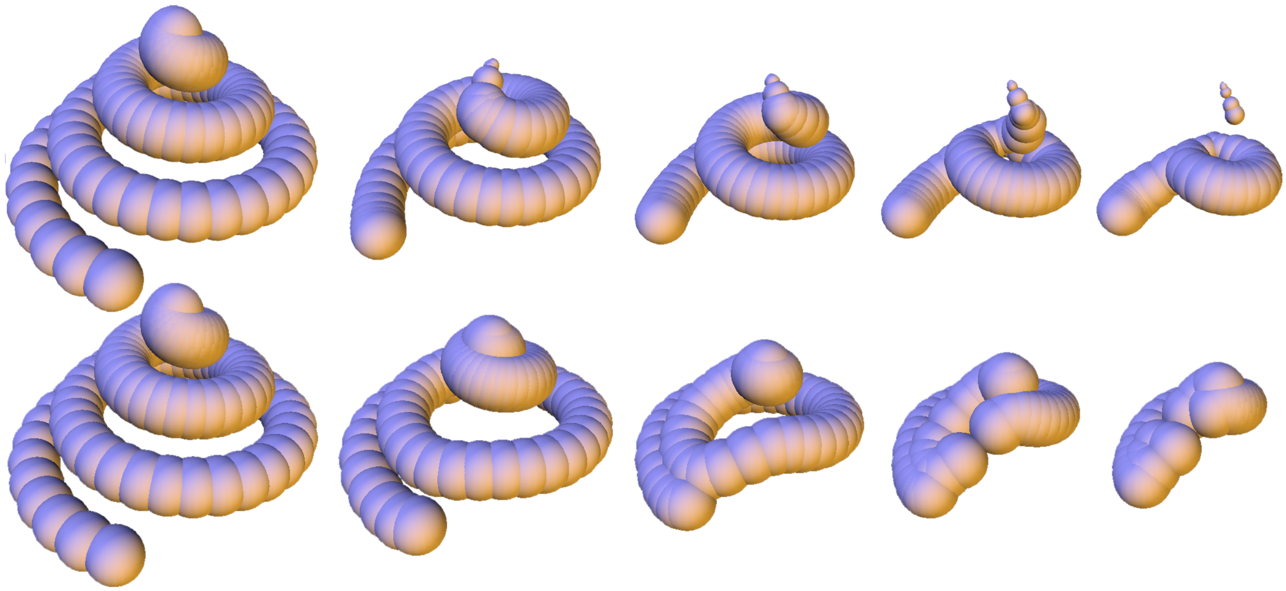


Figure 7. Gaussian filters in space (top) and along the medial axis (bottom) on the same example as Figure 4 fail in avoiding self-intersection.

- *border* when B is adjacent to some triangles and eventually to boundary triangles and singular edges. In that case, the planes are computed considering all pairs of adjacent balls (including triangles) as in the *elbow* case. For each adjacent triangle, we also compute the planes described in the *regular* case.

C. Averaging of planar motions

For each ball B of $\mathcal{U}$ and for each plane P_i computed above, except for the *isolated case*, we compute the intersection of $\mathcal{U}$ with P_i, generating a union of disk $\mathcal{U}_i$. Applying the 2D curvature motion on $\mathcal{U}_i$ results in a new ball B_i whose center belongs to plane P_i or in the removal of the projection of B if it is considered noisy or spurious in $\mathcal{U}_i$. If B is removed in all the planes, it is removed from $\mathcal{U}$. If not, the new position of B is computed from the B_i's as follows.

For the mean curvature-like smoothing, the new ball B^1 is the barycenter of the B_i's. For the minimal (respectively maximal) curvature-like smoothing, B^1 will be the closest (respectively farthest) ball B_i from B. Other motions can be devised, for example a sort of median curvature motion taking for B^1 the ball B_i such that $\|\mathbf{p}_B\mathbf{p}_{Bi}\|$ is the median of the distances $\|\mathbf{p}_B\mathbf{p}_{Bj}\|$, using the difference of ball radius in case of ties (see Figure 11).

Finally, to optimize the process, we do not use the whole intersection $\mathcal{U}_i$ for the 2D motion, since it would lead to a cubic complexity (considering the 2D Voronoï diagram linear and the 3D medial axis quadratic [21]). Instead, for each ball B in $\mathbf{M}$ and each of its planes P_i, we compute the intersections of $\mathcal{U}$ with only balls k-adjacent to B in $\mathbf{M}$ (i.e. adjacent to a ball $(k-1)$-adjacent).

D. Re-sampling

It is well known that, in the differential case, three-dimensional mean curvature motions can cut the original shape [2]. This can be undesirable for smoothing and we can avoid it by imposing a sampling condition on the union of balls, similarly to the 2D case [15]. To do so, we remove two adjacent balls on the medial axis when they are closer than a factor $\epsilon = 0.05$ of their minimal radius. Similarly, when two adjacent balls in $\mathbf{M}$ may disconnect, we insert a new ball at the middle of them, with radius the average of their radius (see Figure 8). To ensure that the shape does not disconnect and to avoid computing the medial axis once more, we use the medial axis from before the movement to the positions of the balls after the movement.

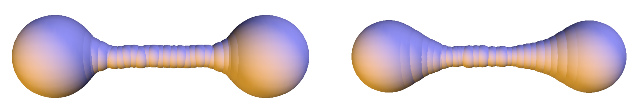


Figure 8. Sampling condition on the dumbbell shape after 3 and 50 iterations.

IV. Experimental Results

The above algorithm has been implemented using the `CGAL` library for the α-shape and the Voronoï diagram [3].

The proposed smoothing of union of balls intends to generate a well-behaved scale space. Actually, the main properties of our approach are inherited from the planar curvature motion: the movement does not create self-intersections (see Figures 4 and 7) ; it smoothly convexifies the shapes (see Figure 1) ; and the shape tends to a ball (see Figure 10). However, similarly to the three-dimensional

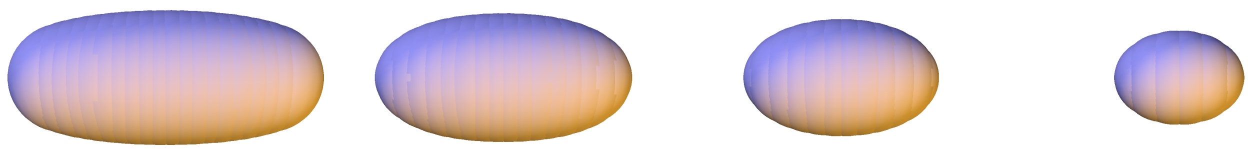


Figure 9. The averaging of local bi-dimensional movements is robust even in completely degenerated cases.

mean curvature motion, the movement can cut the shape. This is a well-known behavior, evidenced on the classical dumbbell shape (see Figure 11(middle)). As experimented in Teixeira's thesis [10], the minimal curvature motion avoids this problem on the dumbbell (see Figure 11(left)). Finally, the maximal curvature motion is much more sensitive to small asymmetries, such as the initial shape (see Figure 11(right)). Cuts can also be avoided by the resampling proposed at Section III-D, which also smoothens the shape contour and turns it more symmetric (see Figure 8).

Degenerate cases may occur for perfectly symmetric shapes, since the medial axis becomes singular, i.e. $\mathbf{M}$ is made only of edges, and colinear centers for the *elbow* case. Even in this case, the proposed method robustly smoothens the union of balls (see Figure 9).

We tested the smoothing on real examples, such as metaballs models for an octopus (see Figure 1) and an RBF model of a fish (see Figure 10), generated from the interior poles of a sparse surface reconstruction [22].

The main limitation of our method is the execution time (see Table I), where the bottleneck is the initial α-shape computation and the actual implementation that re-computes all the structure for each iteration. However, simpler approaches fail to generate a correct scale-space. To exemplify this, we compared with two gaussian filters: a direct filtering in space, moving balls from position B to $\lambda\bar{B} + (1-\lambda)B$, where $\bar{B}$ is the average position of the neighboring balls. This average is weighted by a gaussian function of the distance, and the radius is updated in the same way. The second filter performs the same operation, but computing the distance along the medial axis. This last strategy mainly differs from ours in the movement magnitude of each ball, but the balls generally displace along the normal of the medial axis. We tested those filters on the octopus metaball model (see Figure 1), with $\lambda = 0.05$ and the gaussian function parameter σ set to 10% of the bounding box of the model. We also tested those filters on the spiral model (see Figure 7) with $\lambda = 0.3$ and $\sigma = 30\%$. As compared to our proposal, those simpler movements are much faster (10% of our execution time), but create self-intersections and disconnect the shape.

Table I

SIZES (NUMBER OF BALLS) AND TIMING RESULTS OF SMOOTHING THE MODELS ILLUSTRATED IN THIS PAPER.

Figures	Model	Size	Time per iteration
Figure 1	octopus	103	0.3 s/it
Figure 4	spiral	101	0.7 s/it
Figure 8, Figure 11	dumbbell	21	0.1 s/it
Figure 9	ellipse	20	0.1 s/it
Figure 10	fish	204	1.1 s/it
	sibgrapi	193	0.2 s/it

V. Conclusion

In this work, we propose a smoothing of union of balls by a heuristic based on the medial axis and inspired by bi-dimensional curvature motion. The proposed scheme achieves some of the nice scale-space properties of curvature motion. Moreover, it is robust to degeneracies, and can be easily completed to ensure regular sampling.

As in the bi-dimensional case the averaging at bifurcation points must be weighted, we believe that such results may be improved by a correct evaluation of the weight of each plane. However, this will not influence the smoothing based on the minimal and maximal curvature motions. Another direction for future works would be affine curvature motions for anisotropic smoothing.

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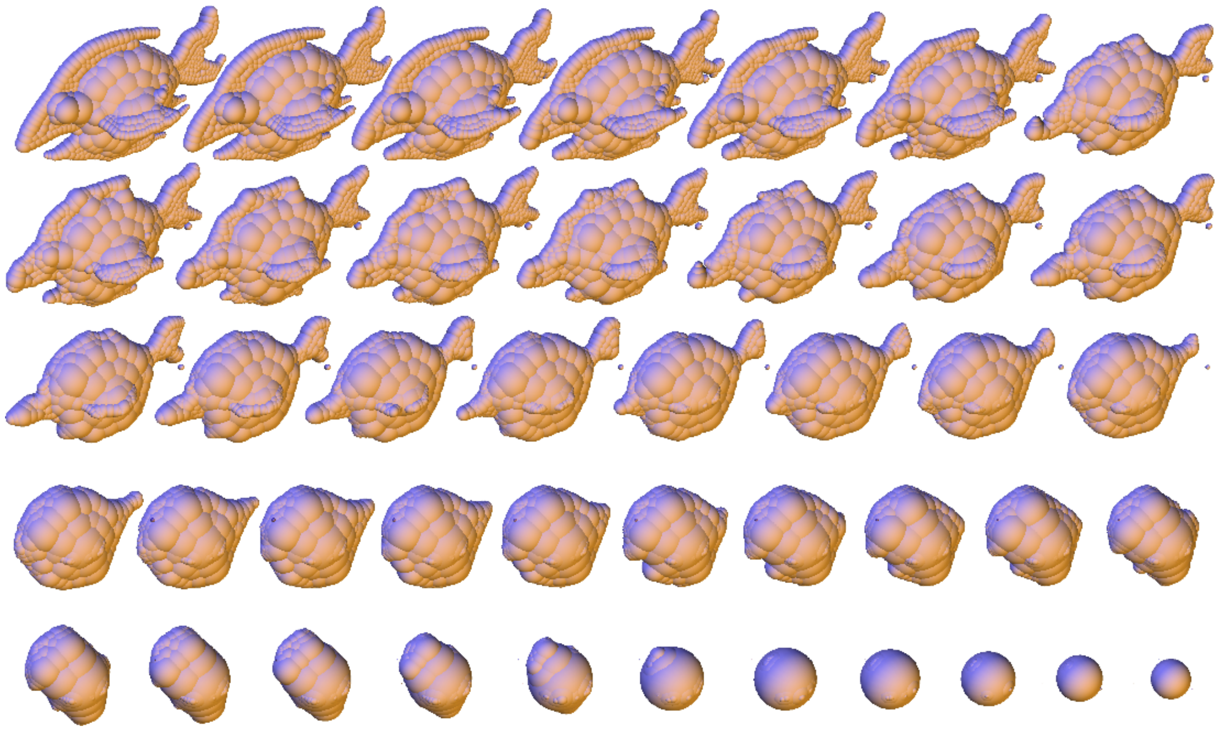

Figure 10. Some of the 300 iterations of the our smoothing applied to an RBF model of a fish, on the positive poles: similarly to curvature motions, our smoothing convexifies the shape in finite time.

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Figure 11. Smoothing a dumbbell shape from min curvature-like motion (left) and mean curvature-like motion (center) and max curvature-like motion (right), after 50, 100, 150 and 200 iterations.

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