On periodic tilings with regular polygons

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*PhD thesis

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***Abstract*—The thesis describes a simple integer-based computational representation for periodic tilings of regular polygons using complex numbers, which is now the state of the art for these objects. Several properties of this representation are discussed, including elegant and efficient strategies for acquisition, reconstruction, rendering, and automatic crystallographic classification by symmetry detection. The thesis also describes a novel strategy for the enumeration and generation of triangle-square tilings via equivalence with edge-labeled hexagonal graphs. The equivalence provide triangle-square tilings with an algebraic structure that allows an unfolding interpretation.**

I. Introduction

Tilings and patterns have been part of human culture for as long as we think ourselves as humans [1]. Their presence is ubiquitous in the human world, be it as ornament in fabric and architecture, or as part of the functional design of materials and structures. Islamic art, with its historical richness of geometric patterns [2], has inspired artists such as Escher and researchers in mathematics and computer science. A method for generating Islamic-style patterns from tilings with regular polygons has been proposed by Kaplan [3]; this is a very direct application of this thesis, among many others.

In mathematical terms, a *tiling* is a subdivision of the plane into bounded closed faces topologically equivalent to a disc. We focus on periodic tilings whose faces are regular polygons and which are *periodic*, that is invariant under two linearly independent translations. These restrictions impose much rigidity while still allowing much variety (Figure 1).

Tiling the plane periodically with regular polygons is an absorbing subject [4], [5]. The oldest formal treatment was proposed 400 years ago by Kepler, in his 1619 book "Harmonices Mundi". Yet, a complete classification of these tilings remains elusive. Lenngren's survey [6] described the previous state of the art for this problem and Chavey [7] made the most recent contribution in the subject until our paper [8], derived from the thesis [9].

Crucial to any computational study or application of tilings is a representation that makes it convenient to synthesize, compare, explore, and analyze tilings. Using standard representations of subdivisions of the plane to represent tilings brings unneeded complexity and numerical problems due to irrational vertex coordinates. Thus, specialized representations are needed. The computational and visualization tools developed as part of the representation turned out to be fundamental to the creation of novel enumeration results and methods.

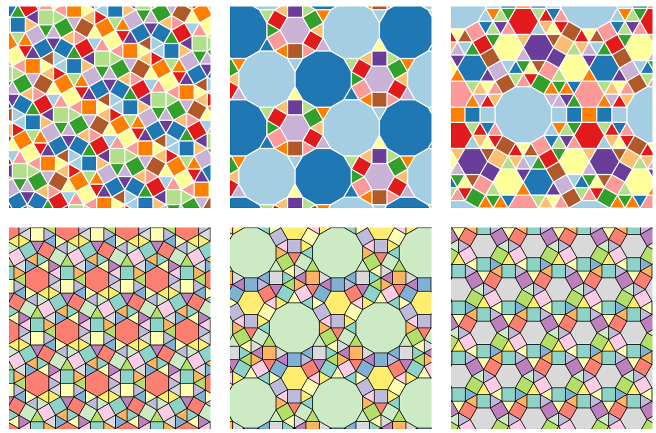


Fig. 1. Examples of periodic tilings of the plane by regular polygons.

The thesis has two main contributions. The first contribution is the formulation of a simple integer-based representation for periodic tilings of regular polygons using complex numbers. This representation allowed us to acquire geometric state-of-the-art models from two large collections of images [10], to synthesize new images of the tilings at any scale with arbitrary precision [11], and to classify tilings by crystallographic groups using symmetry detection. These results are described in Chapters 2, 3, and 4 of the thesis. They are concisely presented in [8] and summarized in Section II.

The second contribution is the proposal and demonstration of a series of results, and the implementation of algorithms that allow the enumeration and generation of periodic *triangle-square* tilings (formed by triangles and squares only). Exploiting the topology of their dual, we derive their equivalence with an edge-labeled hexagonal graph over the flat torus. This allows us to define families of triangle-square tilings and find a set of minimal generators that provide each family with an algebraic structure. This offers a characterization of all triangle-square tilings. We show some experimental results of using this approach to generate tilings and some geometric insight on their algebraic structure. These results are contained in Chapter 5 of the thesis and summarized in Section III.

Three papers have been published from the thesis so far (Section IV). A large collection of tilings in our representation and reference implementations are freely available at the project page: http://chequesoto.info/tilings.html.

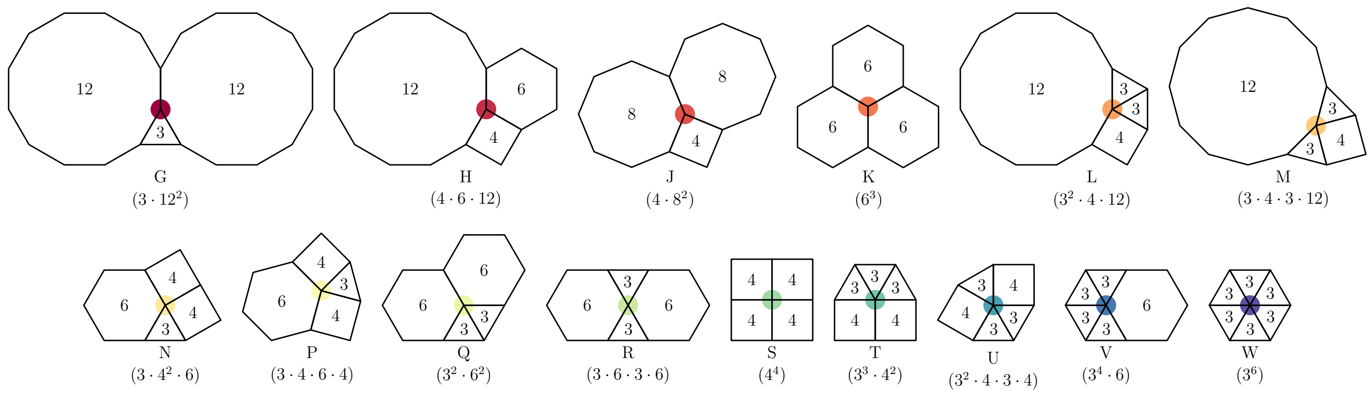


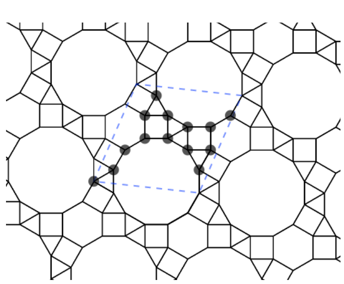
Fig. 2. Archimedean vertices.

II. An integer representation for periodic tilings of the plane by regular polygons

Only a few regular polygons tile the plane. The internal angles of the faces around a vertex must sum 360°. This can be written in terms of the number of sides of the faces around a vertex as $\sum_{i=1}^{p} \frac{k_i - 2}{k_i} = 2$. The 17 integer solutions of this equation translate into 21 polygon configurations around a vertex, but only 15 satisfy global geometric constraints that make the configuration viable. These are known as the *Archimedean* vertices (Figure 2). Thus, all tilings by regular polygons contain only triangles, squares, hexagons, octagons or dodecagons. There is exactly one tiling that includes octagons, the truncated square tiling, composed solely of vertices of type J. Disregarding this singular tiling allows us to align the edges of all the others with the 12th roots of the unity.

A. Lattice coordinates

Our philosophy is to focus on the vertices of a tiling. Without loss of generality, we assume that one of the vertices is at the origin and one of the edges is horizontal. This way, by following paths along the edges, we can represent all vertices of a tiling as integer polynomials in ω, the principal 12th root of the unity. Although a vertex can be reached by different polynomial paths, reducing the polynomials modulo the 12th cyclotomic polynomial $(\omega^4 - \omega^2 + 1)$ guarantees that each vertex is represented as a unique polynomial of degree at most 3, say $a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3$. This allows us to represent each vertex only by its polynomial coefficients $[a_0, a_1, a_2, a_3]$, which we call its *lattice coordinates*.



$$\left[\begin{array}{cccc} 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & -2 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & -1 \\ 2 & 3 & 0 & -1 \\ 2 & 3 & 1 & -1 \\ 2 & 3 & 1 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right]$$

Fig. 3. Example of a tiling and its representation.

The periodicity of the tiling requires that it is invariant under two linearly independent translations t_1, t_2. This creates a grid over the plane in which each cell contains equivalent sets of points. The vertices in the basic cell $T_0 = \{\lambda_1 t_1 + \lambda_2 t_2 : \lambda_1, \lambda_2 \in [0, 1)\}$ are called *seeds*. Thus, every vertex of the tiling is equivalent by translations to exactly one seed. The seeds and their translations decompose the set of vertices into interlocking regular systems of points sharing the same translation group in the sense of Hilbert and Cohn-Vossen [12] (Figure 4, center-left).

Translation vectors are represented in lattice coordinates, since they coincide with vertices equivalent to the one at the origin. This way, a tiling with n seeds is represented by a $(2+n) \times 4$ integer matrix containing lattice coordinates of the translation vectors and the n seeds (Figures 3 and 4 left).

B. Topological primitives

The edges of the tiling are deduced by finding the points at unit distance from each other. This would usually take $\mathcal{O}(n^2)$ distance computations. However, in our representation this can be done in linear time by creating a vertex cloud and a hash table with the lattice coordinates of the seeds and their equivalent vertices: translated copies on the 8 cells around the basic cell. Edges are reconstructed by making a hash query in each basic direction around a seed to check whether there is a vertex. Once the *star* (the pattern of edges around a seed) is computed, faces can be generated procedurally following the basic directions. Applying this strategy in only one orientation for each seed generates a unique copy of each polygonal face, forming a *patch*, which can be propagated to cover an arbitrary region of the plane using the translation grid.

C. Acquisition

At the start of this research, the state of the art in high-order Archimedean tilings were image collections. Algorithms were designed for acquiring the vertices and automatically detecting minimum translation grids. This allowed us to determine the symbols of all exemplars of Galebach collection [13] and all but 3 exemplars of Sá e Sá collection [14].

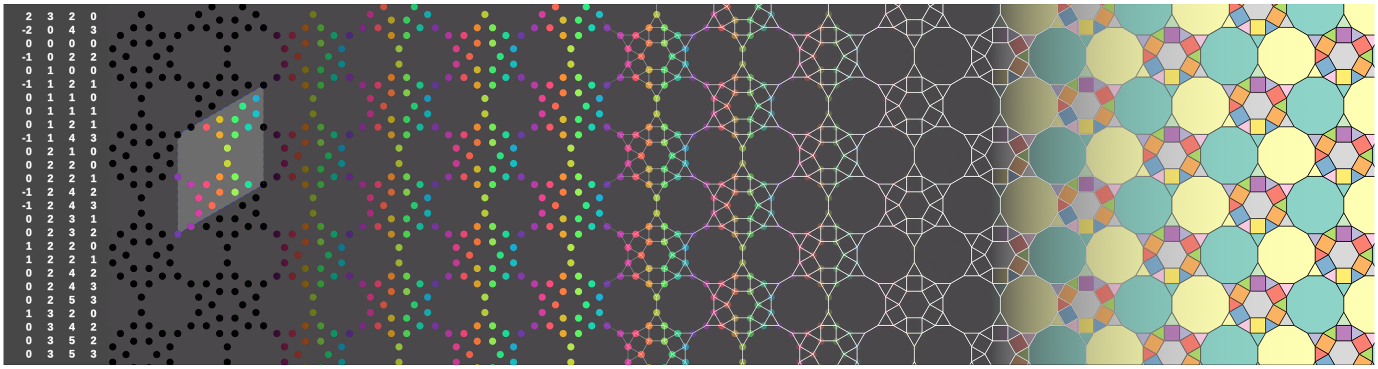


Fig. 4. From left to right: lattice coordinates, fundamental parallelogram and seeds, interlocked systems of points, edges, and full tiling.

D. Symmetry

The symmetries of plane periodic patterns are well understood and have been completely classified: there are exactly 17 symmetry groups, known as the *wallpaper groups* [15]. Our representation allows all symmetry operations to be performed over lattice coordinates. This means that applying simple transformations on the seeds and making hash queries allows the detection of all the symmetries of a tiling. We follow the standard flowchart by Washburn and Crowe [16] to classify the wallpaper groups given the detected symmetries.

A tiling is m-uniform when it has exactly m equivalence classes of vertices under symmetry, and k-Archimedean when the vertices belong to exactly k types of Archimedean vertices. Identifying the vertices inside the crystallographic fundamental region of the tiling – following Schattschneider [17] – allows us to classify a tiling automatically in its m-uniform k-Archimedean classes (Figure 5).

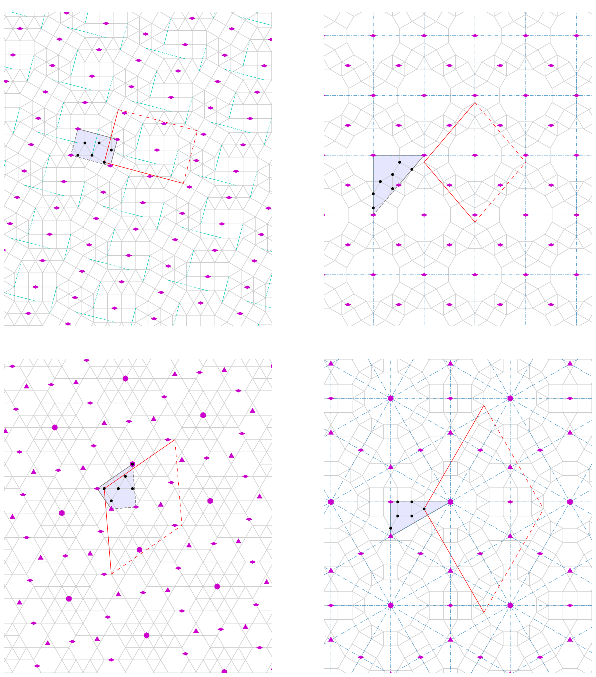


Fig. 5. Tilings with automatically detected and annotated symmetries.

III. Enumeration and Generation of Triangle-Square Tilings

It is natural to ask whether and how one can generate arbitrarily complex tilings. More precisely, can we find, or at least count, all the tilings satisfying some constraints? Constraints could include the area of the fundamental region, the number of vertices or faces, and so on. Is there a set of equations or expressions that determine such tilings? We focus on triangle-square tilings, that is, tilings formed solely by triangles and squares, because these tilings contain all tilings by regular polygons after refinement. Dodecagons are obtained from hexagons with a corona of alternating triangles and squares, and hexagons correspond to groups of 6 triangles.

A. Restricted dual tilings

A triangle-square tiling $\mathcal{T}$ is a combinatorial arrangement of triangles and squares on the plane, which is completely determined by one of two complementary sets of triangles $\mathcal{E}$ and $\mathcal{L}$, present in the tiling. A tiling containing triangles satisfies that either one of these sets is not empty and both have even cardinality.

We define the *restricted dual tilings* $\mathcal{T}_{\mathcal{E}}{}^*$ and $\mathcal{T}_{\mathcal{L}}{}^*$ as the dual tilings on each of these sets, and analyze their combinatorial structure. In the example in Figure 6, one can see the restricted duals and how they correspond to two connected components of $\mathcal{T}^*$, the dual tiling of $\mathcal{T}$.

B. Edge-labeled dual hexagonal graphs

Each restricted dual tiling is isomorphic to a hexagonal graph, as one can observe. We focus on $\mathcal{T}_{\mathcal{E}}{}^*$, but the analysis for $\mathcal{T}_{\mathcal{L}}{}^*$ is analogous, they alternate when a rotation by ω is applied. The number of squares that each edge of the restricted dual crosses can be encoded in the graph, as edge labels. This way, we define $\mathcal{G}_\xi^*$, the *edge-labeled dual hexagonal graph* of $\mathcal{T}$, with labels ξ. Figure 6 illustrates $\mathcal{G}_\xi^*$, the edge-labeled dual hexagonal graph of $\mathcal{T}$, corresponding to $\mathcal{T}_{\mathcal{E}}{}^*$. We draw the graph $\mathcal{G}_\xi^*$ as a hexagonal tiling. The dual of a hexagonal tiling is a triangular one. Thus, the primal $\mathcal{G}_\xi$ of the edge-labeled dual hexagonal graph is a triangle tiling with labels on its edges. Figure 6 illustrates both $\mathcal{G}_\xi^*$ and $\mathcal{G}_\xi$ with their corresponding labels.

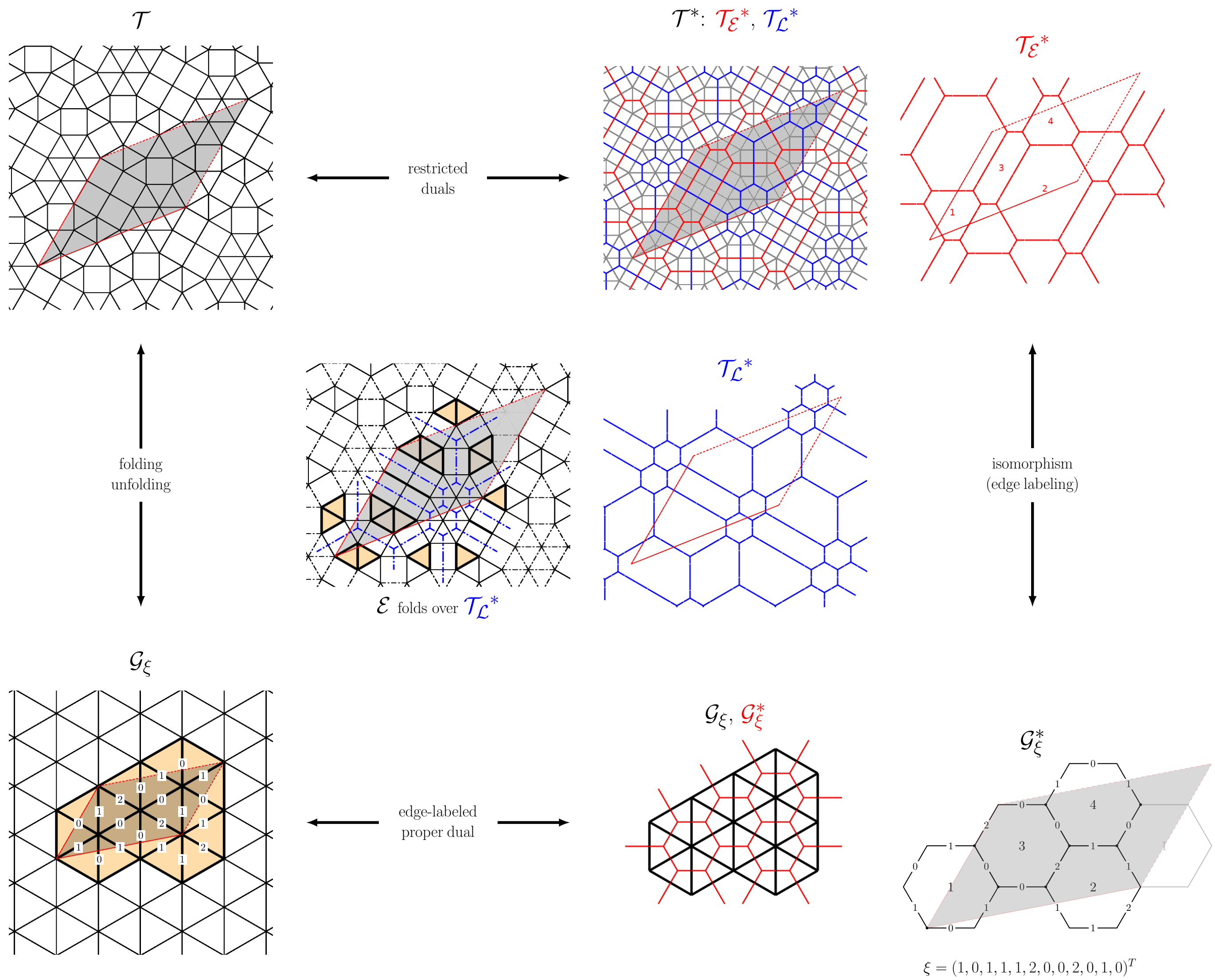


Fig. 6. Relations between a triangle-square tiling $\mathcal{T}$, its restricted duals $\mathcal{T}_{\mathcal{E}}{}^*$ and $\mathcal{T}_{\mathcal{L}}{}^*$, the edge-labeled dual hexagonal graph $\mathcal{G}^*_\xi$ and the folded triangle tiling $\mathcal{G}_\xi$.

$\mathcal{G}_\xi$ is a *folded* version of $\mathcal{T}$, obtained by folding the triangle faces in set $\mathcal{E}$ over the rest of the tiling using $\mathcal{T}_{\mathcal{L}}{}^*$ as the valleys of a *crease pattern*. Figure 6 illustrates $\mathcal{G}_\xi$ obtained by folding $\mathcal{T}$ over the crease pattern with valleys on $\mathcal{T}_{\mathcal{L}}{}^*$. The labels indicate the number of folds under each edge.

C. Enumeration and generation

The enumeration and generation of triangle-square tilings is given by the equivalence of a given tiling $\mathcal{T}$ with its edge-labeled dual hexagonal graph $\mathcal{G}^*_\xi$. This is done by enumerating all possible plane tori tiled with h hexagonal faces, which, in turn, correspond to triangle-square tilings with $2h$ triangles in set $\mathcal{E}$. Over each topology indexed by a pair (i,h), we deduce the geometric constraints that define a valid labeling. These constraints are given by an integer, large and sparse, homogeneous system of equations.

For each family $\mathbb{T}_{(i,h)}$ of tilings in the (i,h) topology, a Hilbert basis for the set of valid labelings is computed by available algebraic geometry methods; in this case *Normaliz* [18] was used. Every valid labeling is obtained as a positive integer linear combination of elements in the basis. Elements of the basis can be interpreted as *inflations* or *unfoldings* of the triangle tiling over discrete geodesics or over discrete 3-fold nets. The positive integer linear combinations form increasingly complex unfoldings, generating new tilings (Figure 7).

Algorithms for the enumeration of the hexagonal graph topologies and for the generation of a triangle-square tiling from a valid labeling are thoroughly discussed in the thesis. Also, complementary conditions between $\mathcal{T}_{\mathcal{E}}{}^*$ and $\mathcal{T}_{\mathcal{L}}{}^*$ are deduced. More than 100 million tilings were generated in order to validate the asymptotic analysis of the behavior of these tilings as the label values grow.

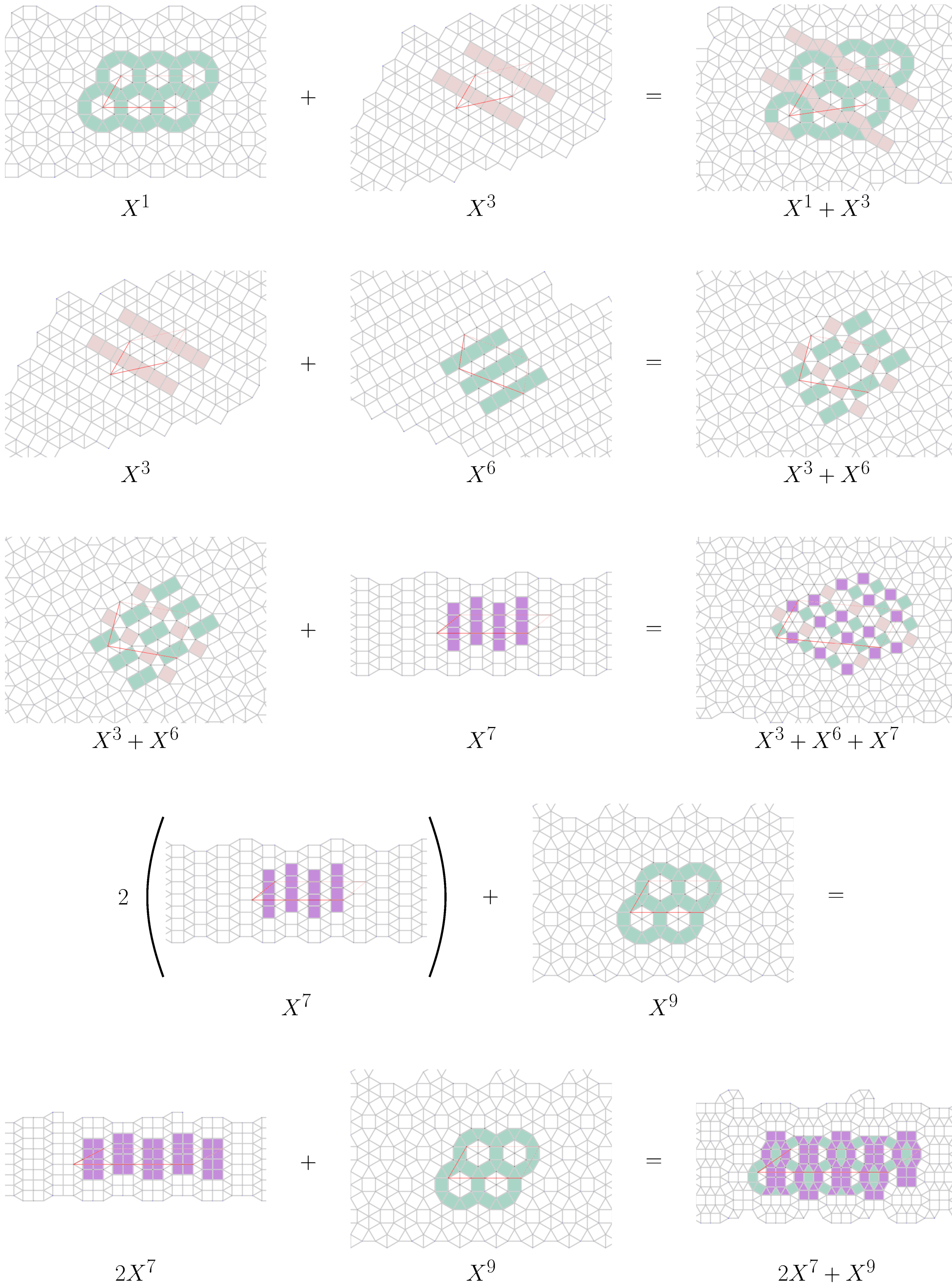


Fig. 7. Examples of operations between tilings corresponding to basic labelings in $\mathbb{T}_{(4,6)}$, the unfoldings are highlighted at each step.

IV. PUBLICATIONS AND FUTURE WORK

Our representation is compact and efficient for rendering, manipulating, and analyzing periodic tilings of the plane with regular polygons. It is the first computational representation of periodic tilings for which there is a large collection of tilings available. Representing these complex mathematical objects with integers allows new robust algorithms for artistic and computational applications. The representation and its properties are the focus of the recent publication *An integer representation for periodic tilings of the plane by regular polygons* [8], which appeared in Computers & Graphics, Vol. 95, 2021. The first results of the research were presented on *Synthesizing periodic tilings of regular polygons* [11] at the 31st Conference on Graphics, Patterns and Images (SIBGRAPI 2018). We also presented *Acquiring periodic tilings of regular polygons from images* [10] at Computer Graphics International 2019, which was published in The Visual Computer Journal, Vol. 35, 2019.

The proposed enumeration and generation strategy for triangle-square tilings is a novel approach to an old established problem. The proposed strategy has different levels of abstraction and establishes connections with other branches of mathematics. It also provides an algebraic interpretation of the *unfolding* or *inflation* phenomenon, clearly observed when analyzing the tiling collections. It has been a great experience tackling a combinatorial geometric problem using a computational approach while having so much visual support from the computer graphics tools, allowing us to enjoy the appealing beauty of the resulting images. Tim Weyrich derived an independent enumeration strategy based on a search over lattice coordinates, and we are currently working together on a publication on tiling enumeration as part of an active research program on periodic tilings with regular polygons.[1]

Apart from the mathematical and computational contributions presented, the thesis has a broad range of applications. I have started exploring extensions of the representation to aperiodic tilings of arbitrary n-fold order. Several ideas and techniques from the thesis can be applied on modeling and digital fabrication on different ways. These are part of a research program that I hope to be able to finance and execute from 2022 onwards. Mathematics popularization is a constant interest for me, I have developed a high precision laser-cut puzzle prototype for its exhibition and distribution on mathematics festivals and museums (Figure 8). The puzzle will be accompanied by a widely varied tiling catalog.

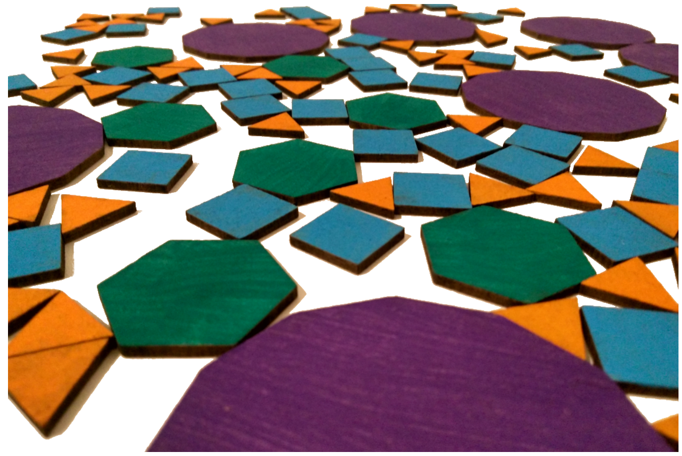


Fig. 8. Tiling puzzle prototype.

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[1]http://chequesoto.info/tilings.html

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