Analysis of optimized quasi-interpolators

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***Abstract*—In this paper we discuss the application of specific constraints to quasi-interpolators, which are a special class of functions used for the continuous reconstruction of an input image when sampled at a given spacing rate. In addition, this research intends to explore some mathematical concepts related to image processing, which are necessary for the comprehension of all the processes along the stages of the traditional sampling framework. This article also focuses on minimizing the residual error of linear schemes, obtained from the optimized quasi-interpolators.**

***Index Terms* - image processing, optimized quasi-interpolators, linear schemes.**

I. Introduction

In computer graphics and applied mathematics, image processing is one of the most important study and research topics, and it usually involves several steps based on some specific mathematical concepts, such as Fourier transform and convolution. For this reason, when performing arbitrary operations on an image (such as rotation, translation, among others), a quick and efficient reconstruction of the input data - in such a way that the error during the process is minimal - is highly desirable. This, in turn, depends on the input information (described by any function), as well as the steps taken throughout the image processing.

In this article the focus is on the so-called optimized quasi-interpolators, which is also a family of reconstruction schemes, recently proposed by Sacht and Nehab [1]. Therefore, there are a few theoretical concepts about image sampling and reconstruction, as well as approximation theory, that will be presented in the next section, so that it is possible to understand the working of the image processing schemes and, consequently, the optimized quasi-interpolators.

A crucial issue in image processing is the conditions under which it becomes possible to reconstruct an image when there is no information about the input function other than just a discrete set of sampled values. In specific cases, it is possible to restore it using the Shannon-Whittaker sampling theorem [2] (a.k.a sinc interpolation). Basically, this theorem is a method to construct a continuous bandlimited function using a special function known as ideal low-pass filter (normalized sinc function): $\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. It states that, when a sample spacing T is known, the initial function f can be transformed to another function g, which is bandlimited to the Nyquist interval: $(-\frac{0.5}{T}, \frac{0.5}{T})$. This process is called pre-filtering:

$$g(x) = \int_{-\infty}^{\infty} f(t) \operatorname{sinc}((t-x)/T)\, \mathrm{d}t. \tag{1}$$

This new function $g(x)$ can be restored using spaced samples $g(kT)$ of itself, $\forall k \in \mathbb{Z}$:

$$g(x) = \sum_{k\in\mathbb{Z}} g(kT) \operatorname{sinc}(x/T - k). \tag{2}$$

However, this theorem only provides perfect reconstruction when the input function is band-limited, which is not the general case. Also, the sinc function has slow decay, which raises the computational cost of the reconstruction. For this reason, the modern sampling framework, illustrated by Figure 1, is constituted by three generalized steps:

- Prefilter ψ stage (a.k.a antialiasing filter): It extends the idea of low-pass filter by using general sampling;
- Digital filter q: A discrete filter which gives more freedom for the choice of the analysis filter ψ;
- Generating function φ, also called reconstruction kernel:

All the steps mentioned in these three items can be described by the equations below:

$$g(x) = \int_{-\infty}^{\infty} f(t)\, \psi((t-x)/T)\, \mathrm{d}t, \tag{3}$$

$$g_k = g(kT), \qquad c_i = \sum_{k\in\mathbb{Z}} g_k\, q_{i-k}, \tag{4}$$

$$\tilde{f}(x) = \sum_{i\in\mathbb{Z}} c_i\, \varphi(x/T - i). \tag{5}$$

The quasi-interpolators assume that the prefilter ψ is unknown (leading to the simplifying assumption $\psi = \delta$, where δ is the Dirac's delta), so all these considerations must be such that all the degrees of freedom (obtained on the steps related to q and φ) are used to minimize the final error and optimize the quasi-interpolators, with emphasis on the usage of linear generators.

II. Notation

Equations (3)-(5) can be shortened using a convenient notation: let $f : \mathbb{R} \to \mathbb{C}$ be an input function and $q : \mathbb{Z} \to \mathbb{C}$, be a digital filter. Function scaling and reflection can be described, respectively, as:

$$f_T(x) = f(x/T) \quad \text{and} \quad f^\vee(x) = f(-x). \tag{6}$$

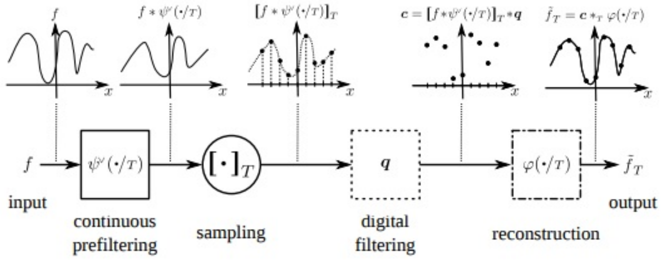

Fig. 1. Modern sampling and reconstruction framework.

Let continuous, discrete and mixed convolutions be denoted by:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, \mathrm{d}t, \tag{7}$$

$$(c * q)_n = \sum_{k \in \mathbb{Z}} c_k\, q_{n-k}, \quad \text{and} \tag{8}$$

$$(c *_T f)(x) = \sum_{i \in \mathbb{Z}} c_i\, f(x - iT). \tag{9}$$

Denote also the discrete convolution inverse as q^{-1} (when it is well defined) to be such as:

$$q * q^{-1} = [\ldots, 0, 0, 1, 0, 0, \ldots] = \delta. \tag{10}$$

Uniform sampling of the input function at a sample spacing T (we consider $T = 1$, when there is no information about it) is given by:

$$[f]_T = [\ldots, f(-T), f(0), f(T), \ldots]. \tag{11}$$

All the definitions developed in (6)-(11) can be used now to describe the reconstructed function $\tilde{f}$ by:

$$\tilde{f} = [f * \psi_T^{\vee}]_T * q *_T \varphi_T. \tag{12}$$

III. Related Work

To obtain efficient reconstruction some authors use concepts related to the digital filtering stage and to the so-called approximation order, which are explored in the next subsection.

A. Previous results

The main problem faced by several authors is related to the idea that the degrees of freedom of φ and q vanish as the application impose specific requirements. In fact, when considering the generating function (5), φ is generally considered as piecewise polynomial with maximum degree N and compact support, stated as W. These choices leads $W \times (N+1)$ degrees of freedom to φ. Another important fact concerns the so-called regularity, a concept which tells that a function $\tilde{f}$ is R times differentiable ($R \in \mathbb{N}$), i.e., $\tilde{f} \in C^R$. It is directly related to the degrees of freedom in φ.

Digital filtering, in turn, was introduced in the modern sampling framework by Hou and Andrews [3], who used functions called B-splines (basically piece-wise polynomials). However, it is important to note that the set of functions which satisfies (5) is a linear subspace of L_2: $V_{\varphi,T} = \{c *_T \varphi_T \mid \forall c \in \ell_2\}$. When looking to a function in $V_{\varphi,T}$ that is as close as possible to the input f, Kajiya and Ullner [4] found the solution $\psi = \varphi$ and $q = [a_\varphi]^{-1}$, where $a_\varphi = \varphi * \varphi^{\vee}$ is the auto-correlation of the generator φ. Their linear optimization problem is such that the orthogonal projection $P_{\varphi,T} f$ satisfies:

$$P_{\varphi,T} f = c^* *_T \varphi_T, \quad \text{where} \tag{13}$$

$$c^* = \arg\min_{c} \|c *_T \varphi_T - f\|_{L_2}. \tag{14}$$

Lastly, the notion of an approximation order $L \in \mathbb{Z}^+$ is related to the measure of the rate T^L when $T \to 0$, i.e., when $\tilde{f} \to f$. Thus, φ has approximation order L when it is the largest integer for which $\exists\, C > 0$ such that

$$\|f - P_{\varphi,T} f\|_{L_2} \leqslant C \cdot T^L \cdot \|f^{(L)}\|_{L_2} \tag{15}$$

$\forall\, f \in \mathbb{W}_2^L$ (the Sobolev space[1]). The orthogonal projection, in turn, can be redefined as:

$$P_{\varphi,T} f = [f * \mathring{\varphi}_T^{\vee}]_T *_T * \varphi_T, \tag{16}$$

where $\mathring{\varphi} = \varphi * [a_\varphi]^{-1}$ is the dual of φ.

Strang and Fix [5], in their work, concluded that φ has approximation order $L \Leftrightarrow V_{\varphi,T}$ contains all polynomials up to degree $L-1$. This consideration about L implies that these polynomials are preserved through the reconstruction, i.e., $\tilde{f}_T = f$, $\forall f \in P_{L-1}$.

It is also possible to quantify the residual as

$$\|f - \tilde{f}\|_{L_2}^2 \approx \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 E(T\omega)\, \mathrm{d}\omega, \tag{17}$$

where $E(T\omega)$ is quantified by Blu and Unser [6], as shown in the next subsection.

An important result derived by Blu et al [7] shows that, given N, W, R, and L, it is possible to describe completely piecewise-polynomial generating functions. Their results use the subset with maximal order and minimum support (MOMS), originated by the minimization of the asymptotic constant C in (15).

A recent article written by Sacht and Nehab [1], however, focused more directly on the reconstruction quality, although most of the previous work have focused on the decay of the asymptotic constant and approximation order.

B. Optimized quasi-interpolators

Based on the previous considerations, specially on Blu and Unser [6] (which obtained the residual to arbitrary ψ, q, and φ), Sacht and Nehab [1] managed to quantify the error between f and $\tilde{f}_T$ (considering L_2 metric), by using:

Theorem 1: $\forall f \in \mathbb{W}_2^r$ with $r > \frac{1}{2}$, the approximation error can be defined by

$$\|f - \tilde{f}_T\|_{L_2} = \left(\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 E(T\omega)\, \mathrm{d}\omega \right)^{\frac{1}{2}} + e(f, T), \tag{18}$$

[1] $f \in \mathbb{W}_2^r \Rightarrow \int_{-\infty}^{\infty} (1+\omega^2)^r\, |\tilde{f}(\omega)|^2\, \mathrm{d}\omega < \infty$.

where $e(f,T)=o(T^r)$ and

$$E(\omega)=1-\frac{|\hat{\varphi}(\omega)|^2}{\widehat{a_\varphi}(\omega)}+\widehat{a_\varphi}(\omega)\left|\hat{q}(\omega)\hat{\psi}(\omega)-\frac{\hat{\varphi}(\omega)}{\widehat{a_\varphi}(\omega)}\right|^2. \quad (19)$$

Proof: See appendix in [6].

In Blu and Unser [6], also, it is showed that the residual has approximation order $L \Leftrightarrow$ all derivatives of E up to order $2L-1$ vanishes at 0, $\forall L>0$.

Considering only the Nyquist interval, Sacht and Nehab [1] rewrote the equation (17) as

$$\|f-\tilde{f}_T\|_{L_2}^2 \approx \int_{-\frac{0.5}{T}}^{\frac{0.5}{T}} |\hat{f}(\omega)|^2 E(T\omega)\,\mathrm{d}\omega, \quad (20)$$

which leads to the following minimization problem:

$$\min \int_{-0.5}^{0.5} |\hat{f}(\omega)|^2 E(T\omega)\,\mathrm{d}\omega. \quad (21)$$

Sacht and Nehab [1] also used the following result:

Theorem 2: Given $W \geqslant N$, $\varphi \in \{N,W,L,R\}$ if and only if there exists a unique set of coefficients $a_{k,\ell}$, $b_{k,\ell}$, and $c_{k,\ell}$ such that

$$\begin{aligned}\varphi(x-\tfrac{W}{2}) = &\sum_{\ell=1}^{M}\sum_{k=0}^{N-L-\ell} a_{k,\ell}\left(\beta_{\mathrm{nc}}^{L+k-1} * \gamma_\ell^{N-L-k}\right)(x)\\ &+\sum_{\ell=0}^{M}\sum_{k=0}^{W-N+\ell-1} b_{k,\ell}\beta_{\mathrm{nc}}^{N-\ell}(x-k) \quad (22)\\ &+\sum_{k=0}^{W-L}\sum_{\ell=0}^{L-R-2} c_{k,\ell}\Delta^{*\ell}\beta_{\mathrm{nc}}^{L-\ell-1}(x-k),\end{aligned}$$

where $M=N-\max(R+1,L)$.

Proof: See Blu et al [7].

In the formulas above, $\Delta^{*\ell}$ and γ_ℓ^n are defined in [7], and

$$\beta_{\mathrm{nc}}^n(x)=\beta^n\left(x-\tfrac{n+1}{2}\right), \quad (23)$$

is the non-centered B-spline.

Given these conditions, the constraints defined by Sacht and Nehab [1] consider the minimization problem in (20), which results in the quasi-interpolators. The optimization problem is defined as follows: given a degree N, solve:

$$\begin{aligned}\underset{q,\mathbf{A},\mathbf{B},\mathbf{C}}{\arg\min}\quad & F(d) := \int_0^d \frac{1}{\omega^2}E(\omega)\,\mathrm{d}\omega \quad (24)\\ \text{subject to}\quad & \varphi \in \{N, N+1, -1, 1\},\\ & \varphi^\vee=\varphi, \quad q^\vee = q,\\ & \int_{-\infty}^{\infty}\varphi(x)\,\mathrm{d}x=1, \quad \sum_{k\in\mathbb{Z}} q_k = 1.\\ & E(0)=0.\end{aligned}$$

Arrays of coefficients $\mathbf{A},\mathbf{B}$ and $\mathbf{C}$ encapsulate the degrees of freedom $a_{k,\ell}, b_{k,\ell}$ and $c_{k,\ell}$ in Theorem 2. In this case it is required first-order approximation $L=1$, the support of φ to be $W=N+1$, the regularity of φ to be $R=1$ and the integration limit to be $d=0.5$. Also, the input function is assumed to have the following relation with ω:

$$\left|\tilde{f}(\omega)\right|^2 \approx \frac{1}{\omega^2}. \quad (25)$$

IV. Constraint and minimization analysis

In this section, the numerical results achieved by Sacht and Nehab [1] are justified by some theoretical analysis. In fact, given the conditions defined on (24) and, consequently, the minimization of the error kernel formulated in (19), it just remains to calculate and analyze the numerical results obtained. Also, we are interested in the case of optimizing a piecewise linear quasi-interpolator ($N=1$).

We first analyze the condition that imposes the integral of φ to be 1. In this case, we also consider symmetry $\varphi^\vee=\varphi$, as well as the equation below, which is obtained from Theorem 2, when setting $L=1$, $W=2$ and $R=-1$ and $N=1$:

$$\varphi(x)=b_{0,0}\beta^1(x)+c_{0,0}\beta^0\left(x+\tfrac{1}{2}\right)+c_{1,0}\beta^0\left(x+\tfrac{1}{2}\right). \quad (26)$$

Applying this constraint to the first equation, we have:

$$\int_{-\infty}^{\infty}\varphi(x)\,\mathrm{d}x = 1 \Rightarrow b_{0,0}+2c_{0,0}=1, \quad (27)$$

Thus:

$$\therefore \quad c_{0,0}=\frac{1}{2}-\frac{b_{0,0}}{2}. \quad (28)$$

Now, regarding the digital filter, we have that $q^\vee=q$ and:

$$\sum_{k\in\mathbb{Z}} q_k = 1 \Rightarrow q_{1,0}+2q_{1,1}+2q_{1,2}=1, \quad (29)$$

In this case:

$$\therefore \quad q_{1,2}=\frac{1}{2}-\frac{q_{1,0}}{2}-q_{1,1}. \quad (30)$$

Finally, the third requirement is concerned with the error kernel:

$$E(0)=0 \Rightarrow \frac{(-b_{0,0}-2c_{0,0}+q_{1,0}+2q_{1,1}+2q_{1,2})^2}{\left[q_{1,0}+2\left[q_{1,1}+q_{1,2}\right)\right]^2}=0, \quad (31)$$

So, the equation given by (31) leads to:

$$\therefore \quad q_{1,2}=\frac{b_{0,0}}{2}+c_{0,0}-\frac{q_{1,0}}{2}-q_{1,1}. \quad (32)$$

An interesting fact about these considerations can be seen when we substitute (28) into (32), which leads us to the equation (30). Thus all the five equations given in (24) can be combined into two independent equations.

Now, as seen in Theorem 1, the approximation error basically depends on φ and q, which were constructed according to the optimization constraints. So, with the equations (27)-(31), we can now rewrite the error kernel as:

$$\begin{aligned}E(\omega)=1+&\frac{3+b_{0,0}^2+(3-b_{0,0}^2)\cos(2\omega\pi)}{6\left[q_{1,0}+2q_{1,1}\cos(2\omega\pi)+(1-q_{1,0}-2q_{1,1})\cos(4\omega\pi)\right]^2}-\\ &\frac{2\sin\omega\pi\left[(1-b_{0,0})\omega\pi\cos(\omega\pi)+b_{0,0}\sin(\omega\pi)\right]}{(\omega\pi)^2\left[q_{1,0}+2q_{1,1}\cos(2\omega\pi)+(1-q_{1,0}-2q_{1,1})\cos(4\omega\pi)\right]}\end{aligned} \quad (33)$$

Once the error kernel (33) is known, the objective now is to minimize $F(d) = F(0.5)$ in (24). Unfortunately, the integrand $\frac{1}{\omega^2}E(\omega)$ could not be integrated algebraically, even using the software Wolfram Mathematica. However, a numerical optimization framework is an alternative for this issue. In fact, approximating the integrand by an interpolating polynomial and integrating it leads to values of $b_{0,0}$, $q_{1,0}$ and $q_{1,1}$ that minimize $F(0.5)$. The same results were found by Sacht and Nehab [1], when considering a linear quasi-interpolator ($b_{0,0} = 0.79076352$, $q_{1,0} = 0.77412669$ and $q_{1,1} = 0.11566267$). We now show that this result obtained numerically indeed corresponds to a local minimum of (24). Remind that we have already substituted all constraints into the objective function, which turns this problem into a ultra-constrained minimization one.

As we mentioned in this section, $F(0.5)$ and the error kernel now depends only on $b_{0,0}$, $q_{1,0}$ and $q_{1,1}$, so we can redefine then as $F(0.5) = F(b_{0,0}, q_{1,0}, q_{1,1})$ and $E(\omega) = E(b_{0,0}, q_{1,0}, q_{1,1}, \omega)$, i.e.:

$$F(b_{0,0}, q_{1,0}, q_{1,1}) := \int_0^{0.5} \frac{1}{\omega^2}\, E(b_{0,0}, q_{1,0}, q_{1,1}, \omega)\, \mathrm{d}\omega. \quad (34)$$

To show that $(0.79076352, 0.77412669, 0.11566267)$ is a local minimum, we must first prove that it is a critical point (by analyzing the gradient of F, which must be 0), as we can see below:

$$\nabla F(b_{0,0}, q_{0,0}, q_{1,1}) := \left(\frac{\partial F}{\partial b_{0,0}}, \frac{\partial F}{\partial q_{1,0}}, \frac{\partial F}{\partial q_{1,1}}\right), \quad (35)$$

$$\nabla F(0.79076352, 0.77412669, 0.11566267) = \left(-2.65722 \times 10^{-8}, -4.57467 \times 10^{-9}, 1.14989 \times 10^{-8}\right). \quad (36)$$

Since all the three entries in (36) are almost 0, we can assume that $\nabla F \approx 0$ at this point, which leads us to conclude that the considered point is really close to a critical point. Now, for it to be in fact a local minimum, the Hessian matrix must be a positive-definite matrix, which can be confirmed when its leading main minors are all positive. The Hessian matrix is defined as:

$$\mathrm{H}\left[F(b_{0,0}, q_{0,0}, q_{1,1})\right] := \begin{bmatrix} \frac{\partial^2 F}{\partial b_{0,0}^2} & \frac{\partial^2 F}{\partial b_{0,0}\partial q_{1,0}} & \frac{\partial^2 F}{\partial b_{0,0}\partial q_{1,1}} \\ \frac{\partial^2 F}{\partial q_{1,0}\partial b_{0,0}} & \frac{\partial^2 F}{\partial q_{1,0}^2} & \frac{\partial^2 F}{\partial q_{1,0}\partial q_{1,1}} \\ \frac{\partial^2 F}{\partial q_{1,1}\partial b_{0,0}} & \frac{\partial^2 F}{\partial q_{1,1}\partial q_{1,0}} & \frac{\partial^2 F}{\partial q_{1,1}^2} \end{bmatrix} \quad (37)$$

Thus, the determinant of H in (37) and the principal minors in the considered point are:

$$\mathrm{H}\left[F(0.79076352, 0.77412669, 0.11566267)\right] = \begin{bmatrix} 4.38047 & -4.75701 & -4.75701 \\ -4.75701 & 48.5568 & 46.7768 \\ -4.75701 & 46.7768 & 95.6997 \end{bmatrix} \quad (38)$$

$$\Rightarrow \det(\mathrm{H}) = 9623.35 > 0, \quad (39)$$

$$\det \begin{bmatrix} 4.38047 & -4.75701 \\ -4.75701 & 48.5568 \end{bmatrix} = 190.07 > 0, \quad (40)$$

and, obviously, $\det(4.38047) > 0$, which proves that the analyzed point is a local minimum.

V. Conclusion and Future Work

In this work, we have presented a theoretical analysis of linear optimized quasi-interpolators, which show better results in the residual error minimization when compared to other reconstruction schemes.

Due to the algebraic complexity found in the kernel error analysis we will consider in the future other alternatives to prove that the parameters studied in this article also generate a global minimum for $F(0.5)$. Regarding reconstruction schemes with higher degrees (quadratic, cubic, ...), we will also use the methods explored in this work to optimize reconstruction quality.

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