Multi-scale Morphological Image Simplification Based on Extrema Relationships: Improvements and Applications

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***Abstract*—Image simplification has been proved useful in several image processing applications as an additional step for more complex tasks, such as segmentation and feature extraction. In this work, we explore a graph-based simplification method that guarantees a well-behaved suppression of the image extrema (maxima/minima) by taking into account both information of distance and contrast, as well as some interesting aspects of the scale-space theory. By highlighting some new properties of the method, we define a local update of the graph which implies in an interesting bypass in the whole algorithm structure which, originally, is very time-consuming. Finally, we illustrate how to combine this simplification process with well-known morphological tools to approach problems related mainly with multi-scale image segmentation and homogenization.**

***Keywords*-image simplification; scale-space; mathematical morphology; regional extrema;**

I. Introduction

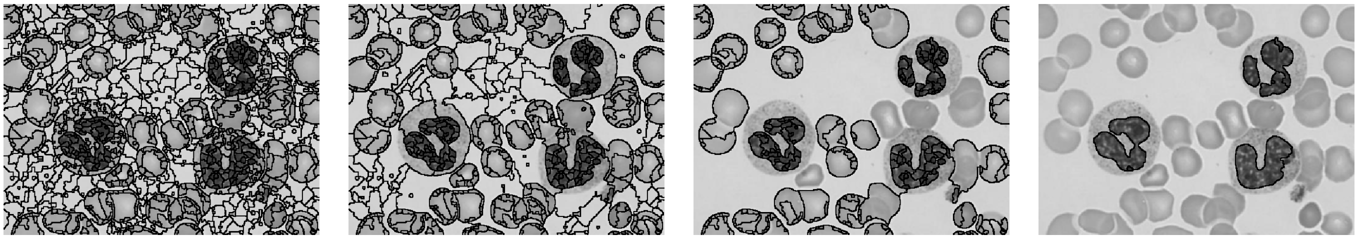
The area of image processing and analysis includes a large amount of applications ranging from the lowest level tasks (e.g. extremum points detection) to the more specialized ones, such as segmentation and classification, requiring the deletion of unnecessary details of the input images. In such a case, it is common to apply a pre-processing step to the original image before any further consideration. Often, this step aims not only to filter out noisy components, for instance, but also to simplify the image through the elimination of non-significant details, while keeping the information necessary to the achievement of the desired outcome.

Image simplification is commonly referred to as a pre-processing step and, in particular, Mathematical Morphology [1],[2],[3],[4] introduces interesting low-level simplification filters exhibiting well-known properties [5]. The typical filtering by opening and closing [2] and their combination as alternate sequential filters are commonly used to eliminate undesirable components of an image while preserving its main content. The multi-scale approach based on the scale-space theory [6] defines a multi-level processing (from finer to coarser scales) related to different representations of the original signal. In such a case, the simplification should be well-behaved in the sense that, given a certain feature of interest (e.g., the zero-crossing of a function), one seeks to track its representation along the different scales. This multi-level transformation should satisfy some properties of $monotonicity, continuity$, $fidelity$ and $euclidean\ invariance$ [6]. The monotonicity concept guarantees the non-inclusion of new interest features at different scales; the continuity states that a continuous path should be defined by the remaining features along these scales; the fidelity ensures that the signal tends to its original form as the scale tends to zero, and, finally, the euclidean invariance asserts that translation and rotation transformations result in translated and rotated signals.

Morphologically, the leveling approach in [7],[8] defines a reconstruction-based simplification without changes in the final contours w.r.t the original image. Another example of morphological simplification is the dynamic measure [9] which selects components of an image according to the notion of extinction values (e.g., area or volume) [10]. This procedure is closely related to the measure of persistence of a signal and is used to eliminate image components regarded as non-significant. Another method based on scale-dependent non-flat structuring function is discussed in [11], where a toogle-like transformation simplifies an image in a self-dual way.

Recently, the work in [12] introduced a non-self-dual simplification method taking into account the relationship between image extrema. More specifically, it considers the distance and contrast between the various regional maxima (minima) and define a total order relation closely linked to the degree of simplification one wants to impose. As we will see later, this multi-level process establishes a non-decreasing and well-behaved activity from which the least significant extrema merge successively with the most significant ones, in a given neighborhood.

In this work, we explore this graph-inspired simplification process aiming at improvements in terms of algorithm and applications. We highlight some new properties concerned with the local update of the graph configuration that yields, among others, a huge speed-up in execution time. We also propose new means to combine this algorithm with different morphological tools and explore the homogenization aspect led by the well-behaved merging of our features of interest (the image extrema). An overview of our work can be seen in Fig. 1 and Fig. 3. As it will become clear later, in Fig. 1, we show some tuples of the aforementioned order relation. These tuples were used here to define a meaningful segmentation of



(a) $\sigma = 24.5$ (b) $\sigma = 4.58$ (c) $\sigma = 1.67$ (d) $\sigma = 0.03$

Fig. 1. Monotonic reduction of image extrema with final watershed segmentation.

the image represented mostly by its deepest regional minima. Fig. 3 shows the bypass introduced here to approach the high computational cost of the original algorithm. As we will see in Section III, this bypass is given by a local update of the graph and concerns the different scale parameters defined along the successive simplifications.

The rest of this paper is organized as follows: Section II presents an overview of the considered simplification algorithm. Section III underlines new properties of this graph-inspired approach and introduces some results leading to an improvement of the algorithm in terms of execution time. Application examples combining morphological tools with the corresponding simplification process are given in Section IV. Finally, some conclusions are drawn in Section V together with future works on this matter.

II. Background

The main concepts behind the simplification process explored in this work consider the notion of scale-dependent erosion and dilation, as stated in [13], [14] and further explored, for example, in [11],[12]. These operations can be defined as follows [13].

Definition 1 (Erosion)**.** *The erosion of the function $f(x)$ with the structuring function $g_\sigma(x)$, noted $[\varepsilon_{g_\sigma}(f)](x)$, is defined by:*

$$[\varepsilon_{g_\sigma}(f)](x) = inf_{t \in \mathcal{G} \cap \mathcal{D}_{-x}} \{f(x+t) - g_\sigma(t)\} \qquad (1)$$

Definition 2 (Dilation)**.** *The dilation of the function $f(x)$ with the structuring function $g_\sigma(x)$, noted $[\delta_{g_\sigma}(f)](x)$ is defined by:*

$$[\delta_{g_\sigma}(f)](x) = sup_{t \in \mathcal{G} \cap \check{\mathcal{D}}_x} \{f(x-t) + g_\sigma(t)\} \qquad (2)$$

where $f : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}$ is the image function, $\mathcal{D}_x$ is the translate of $\mathcal{D}$, $\mathcal{D}_x = \{x + t : t \in \mathcal{D}\}$, and $\check{\mathcal{D}}_x$ is the reflection of $\mathcal{D}$. Finally, $g_\sigma : \mathcal{G}_\sigma \subset \mathbb{R}^2 \to \mathbb{R}$ is the scaled structuring function. An example of such a function is given in [14] where $g_\sigma(x) = -\frac{1}{2\sigma}x^2$, and $\sigma > 0$ is the scale parameter.

The result of these operations depends on the origin of the structuring function g_σ. To avoid level shifting and horizontal translation effects, respectively, one must consider [13]:

$$sup_{x \in \mathcal{G}_\sigma}\{g(x)\} = 0 \quad \text{and} \quad g(0) = 0 \qquad (3)$$

It was also proved in [13] that the scale-space conditions ensuring no changes in the original gray-scale and position of the remaining extrema, as well as the non-introduction of new extrema in the simplified signal, are also obtained from this type of structuring function whose shape is concave downward and monotonic decreasing along any radial direction from its origin [13].

In this work, we use the non-flat pyramid-shaped structuring function shown in Fig. 2 [11], and further discussed in Section II-B .

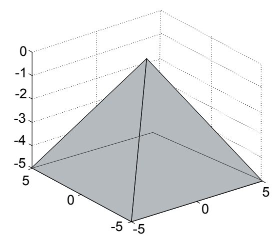


Fig. 2. Non-flat pyramid-shaped structuring function.

A. Extrema relationship

Let $G = (V, E, f)$ be a connected labeled simple graph representing a signal $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$. The set of edges $E \subseteq V \times V$ describes the connectedness of the signal over the vertices V. This graph provides a structure for the function f and is used to simplify this function by means of the suppression of regional extrema. The following definitions concern the scale parameter and the order relation which guarantees a well-behaved transformation, in the sense that no new extrema are introduced in the process [12].

Definition 3. *Let $G = (V, E)$ be a graph and let $u, v \in V$. Let $P(u, v)$ be the set of vertices along one of the paths of shortest length between u and v. Then u and v are said to be ℓ'-separated by $\ell'(u, v)$, with $\ell'(u, v) = \#(P(u, v) - \{u, v\})$.*

Definition 4. *Let $G = (V, E)$ be a graph and let $E_A \subseteq E$ and $E_B \subseteq E$ be subsets from which the sets A and B are built such that A contains all the vertices that form the edges in E_A, respectively for B, with $A \cap B = \emptyset$. Then A and B are said to be ℓ-separated by $\ell(A, B)$, with $\ell(A, B) = min\{\ell'(u, v) | \forall u \in A, \forall v \in B\}$.*

Definition 5. *Given the same conditions as in Definition 4 then A and B are s-neighbors if $\ell(A, B) = 1$.*

Fig. 4 illustrates these definitions, where the two regional minima X and Y, in the solid rectangles, are $\ell\text{-}separated$ by

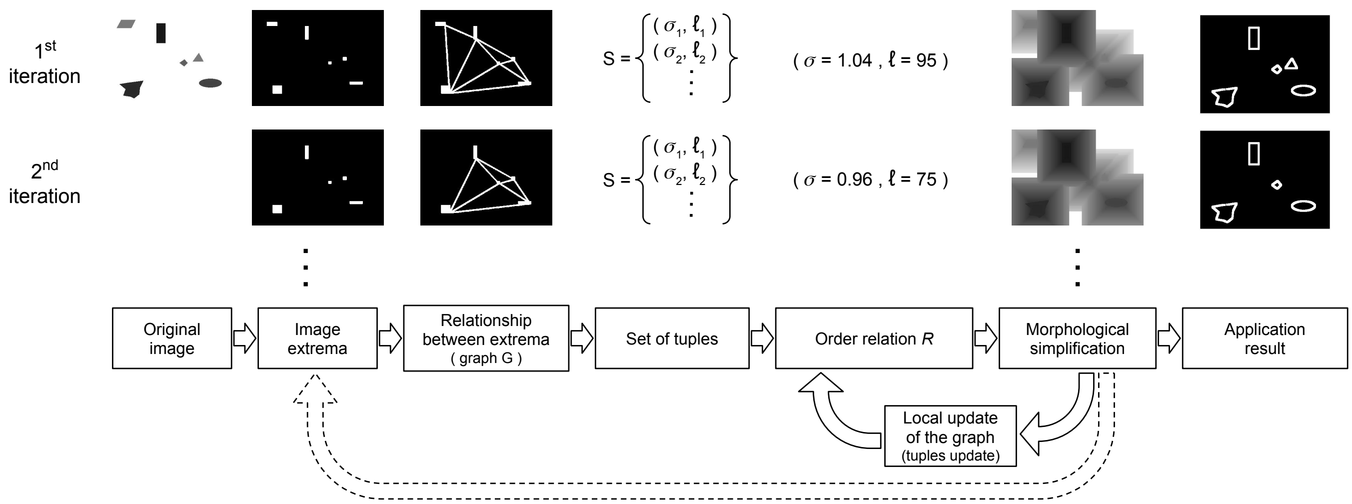


Fig. 3. The original algorithm and the bypass obtained from a local update of the graph.

$\ell(X, Y) = 3$. Now, suppose any function z (e.g., the watershed transform) producing the sets A and B from X and Y (the closed dashed curves), such that $X \subseteq A$ and $Y \subseteq B$ with $\ell(z(X), z(Y)) = \ell(A, B) = 1$. Note that vertices outside sets A and B (nodes labeled 5, 6, 7) constitute the water divisors of function z [15], [16]. This function plays an important role in the simplification process, since it defines, based on sets A and B, where two minima (maxima) are s-$neighbors$, thus reducing the amount of connections generated among extrema.

The proposed relation taking into account both contrast and separation between image extrema can be obtained as follows [12]. Let X and Y be, as before, two extrema of a signal f (both minima or maxima), and $u \in X$, $v \in Y$. Also, consider a function z so that the sets given by $z(X, Y)$ are s-$neighbors$. Then, the following equation defines the height (scale) of the structuring function used in the simplification process.

$$\sigma(X, Y) = \frac{|f(u) - f(v)|}{\ell(X, Y)} \tag{4}$$

By considering Equation (4) for every pair of s-$neighbor$ sets, one can obtain a meaningful collection of tuples associated with the height, σ, and the separation, ℓ, of a pair of extrema. Now, let $S = \{(\sigma_1, \ell_1), \ldots, (\sigma_n, \ell_n)\}$ represent the set of all tuples obtained from these extrema. Then, a strict total order relation R on S, between two tuples, can be defined as follows:

$$(\sigma_p, \ell_p) \prec (\sigma_o, \ell_o) \Longleftrightarrow \sigma_p > \sigma_o \vee (\sigma_p = \sigma_o \wedge \ell_p < \ell_o) \tag{5}$$

This relation guides the way the orderly simplification occurs by successively suppressing the signal extrema, from the least significant to the most relevant, as explained next.

B. Signal Simplification

The simplification process starts by choosing a tuple $t = (\sigma_t, \ell_t) \in S$. Here, the value σ_t is used to define the corresponding structure function $g_t : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ [12]. This function, whose height is given by Equation (4), is connected according to the graph of the original signal and its size extends only to the neighbors of a vertex v. The structuring function g_t is defined by

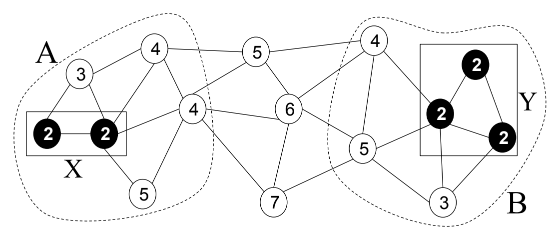


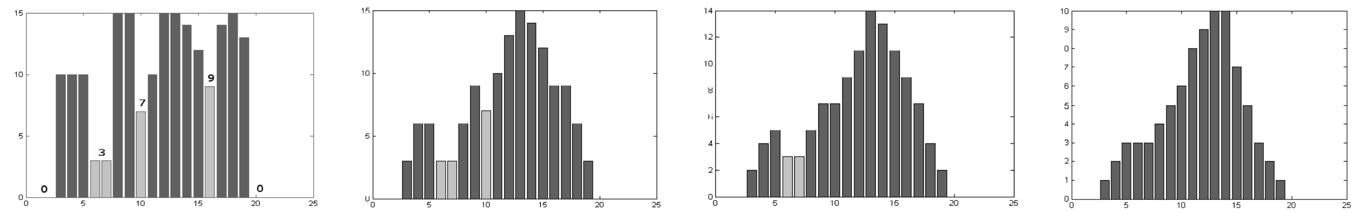
Fig. 4. Graph G of a signal f. X,Y are regional minima in A,B as s-$neighbors$.

$$g_t(x) = \begin{cases} 0 & \text{if } x = 0 \\ -\sigma_t & \text{otherwise} \end{cases} \tag{6}$$

Note that g_t respects $sup(g_t) = 0$ and $g_t(0) = 0$, as required (Equation (3)). In terms of implementation, for a rectangular 8-connected grid, g_t is an elementary $3{\times}3$ discretized pyramid, as in Fig. 2, obeying the scale-space properties mentioned earlier. Through this structuring function, the signal can be simplified by either performing erosions (for regional minima) or dilations (for regional maxima).

Finally, in order to impose restrictions over which extrema are suppressed, at each scale σ_t, one needs to limit the simplification solely to those regional minima, R_{min}, (maxima, R_{max}) influenced by a $t \in S$. The operator τ, yielding the orderly simplification of the regional minima, can be expressed as follows:

$$\hat{f}(x) = \begin{cases} f(x), & \text{if } x \in R_{min}(f) \\ \infty, & \text{otherwise} \end{cases} \tag{7}$$



(a) Signal f (b) $\sigma = 3.0 \;\; \ell = 3$ (c) $\sigma = 2.0 \;\; \ell = 2$ (d) $\sigma = 1.0 \;\; \ell = 3$

Fig. 5. A signal f followed by the merging of its regional minima according to the tuples t in R.

$$\tau(f,t) = inf\left(\varepsilon\left(\hat{f}, g_t\right)^{\ell_t}, f\right) \tag{8}$$

By duality, the simplification of the regional maxima is given by:

$$\breve{f}(x) = \begin{cases} f(x), & \text{if } x \in R_{max}(f) \\ -\infty, & \text{otherwise} \end{cases} \tag{9}$$

$$\tau(f,t) = sup\left(\delta\left(\breve{f}, g_t\right)^{\ell_t}, f\right), \tag{10}$$

where ℓ_t in tuple t indicates the number of erosions/dilations performed with the elementary structuring function g_t described in the Equation (6). This number of iterations is concerned with the following result in [12]

Proposition 1. *if a tuple $t \in S$ is seleted, then at most ℓ_t erosions (dilations) with the structuring function g_t are needed to suppress a minimum (maximum) using $\tau(f,t)$*

Fig. 5 illustrates this simplification process, where the original signal in Fig. 5(a) has five regional minima. Fig. 5(b)-(d) show the resulting simplifications where a less significant minima merges successively with a more significant, according to the order of the tuples in R. These different configurations are obtained by Equation (8) which uses erosion when dealing with regional minima. Thus, based on (4), the tuple t = (3.0, 3), for example, is defined for the two right-most regional minima where, according to (6), the structuring function is given by the one-dimensional discrete function $g = [-3, 0, -3]$. From Proposition 1 above, $\ell = 3$ is the number of erosions necessary for the merging of the two right-most extrema, independently of the signal values between them.

III. New theoretical results

Fig. 3 depicts the main steps of the algorithm described by Equations (8) and (10) above. First, a graph G expressing the neighborhood connectedness of the regional maxima (minima) is defined. Further, the set $S = \{(\sigma_1, l_1), \ldots (\sigma_n, l_n)\}$ of tuples is defined and an order relation R is obtained for the current simplification step represented, for instance, by the highest tuples generating very few simplification activity.

Since we reiterate this whole process, from the least to the most significant extrema, the main drawback of this algorithm is its high computational cost given by the graph construction and tuples sorting.

The new results discussed next explore the graph-inspired implementation of the method together with its monotonicity and continuity properties. In our case, these features ensure that no new extrema are created and that the remaining extrema are not displaced along the different scales. As explained elsewhere, these results will allow the bypassed connection defined by a local update of the graph, as illustrated in Fig. 3.

The next proposition considers the local update of the graph after a least significant extremum merges with one of its neighbors (the most significant in the connected neighborhood). This update defines the new tuples between the remaining vertices and can generate more than one path for a pair of different nodes. In such a case, the order relation R defines the tuple exhibiting the shortest distance between these nodes.

Let X, Y, W be three regional minima (maxima) of a discrete signal. Also, let $s \in X$, $u \in Y$ and $v \in W$ be any point of these extrema belonging to two shortest paths $P(s,v)$ and $P(u,v)$, as illustrated in Fig. 6. From Definition 3, we have that $P(s,v)$ is the set of vertices along one of the shortest paths between s and v of length $\ell'(s,v) = \#(P(s,v) - \{s,v\})$.

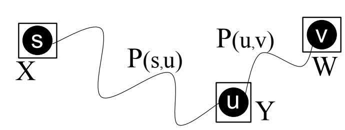


Fig. 6. Shortest paths between regional extrema.

Proposition 2. *If $s \in X$, $u \in Y$ and $v \in W$ are nodes in the regional extrema X, Y and W, where $\{s,u\}$ and $\{u,v\}$ are $\ell' - separated$ respectively by $\ell'(s,u)$ and $\ell'(u,v)$, and u is a node in the least significant extremum Y, then the separation between the remaining extrema, containing s and v, is given by $\ell'(s,v) = \ell'(s,u) + \ell'(u,v) + 1$.*

Proof:

From Definition 3, we have that:

$$\begin{aligned} \ell'(s,v) &= \#(\; P(s,v) - \{s,v\} \;) \\ \ell'(s,v) &= \#P(s,v) - \#\{s,v\}, \end{aligned} \tag{11}$$

where $\#P(s,v)$ can be given simply by

$$\#P(s,v) = \#P(s,u) + \#P(u,v) - \#\{u\}, \tag{12}$$

since node u was considered twice.

Now, replacing Equation (12) into (11) yields:

$$\begin{aligned}
\ell'(s,v) &= \#P(s,u) + \#P(u,v) - \#\{u\} - \#\{s,v\} \\
\ell'(s,v) &= \#(\ P(s,u) - \{s,u\}\ ) + \#\{s,u\} + \\
&\quad \#(\ P(u,v) - \{u,v\}\ ) + \#\{u,v\} - \\
&\quad \#\{u\} - \#\{s,v\} \\
\ell'(s,v) &= \ell'(s,u) + \ell'(u,v) + \#\{s,u\} + \\
&\quad \#\{u,v\} - \#\{u\} - \#\{s,v\} \\
\ell'(s,v) &= \ell'(s,u) + \ell'(u,v) + 1 \qquad (13)
\end{aligned}$$

■

Proposition 3 gives the new $\ell-$separation between the remaining extrema X and W, which are now connected in the graph after the suppression of Y as a regional extremum. This proposition can be easily extended to any number of more significant extrema connected to Y which, in turn, will merge with the most relevant one (according to the current tuple $t \in S$).

Proposition 3. *If X, Y and W are regional extremum vertices ℓ-separated by $\ell(X,Y)$ and $\ell(Y,W)$, and Y is the least significant extremum in the neighborhood, then the update of the ℓ-separation between X and W, after Y merges with one of them, is given by $\ell(X,W) = \ell(X,Y) + \ell(Y,W) + 1$*

Proof:

From Definition 4, we have that:

$$\ell(X,W) = min\{\ell'(s,v) | \forall s \in X, \forall v \in W\}$$

and from Proposition 2:

$$\begin{aligned}
\ell(X,W) &= min\{\ell'(s,u) + \ell'(u,v) + 1\} \\
\ell(X,W) &= min\{\ell'(s,u)\} + min\{\ell'(u,v)\} + 1 \\
\ell(X,W) &= \ell(X,Y) + \ell(Y,W) + 1 \qquad (14)
\end{aligned}$$

■

Now, let us consider the case illustrated in Fig. 7 where a given tuple leads the merging, at the same time, of a set of least significant extrema with another vertice in their neighborhood. For the sake of simplicity, we consider in the following proposition the merging of only two extrema (labeled Q and Y). The given result can be easily extended to any number of nodes.

Let X, Y and Q be the regional minima (maxima) of a signal f. Also, let $s \in X$, $k \in Q$ and $u \in Y$ be any point of these extrema, belonging to one of the shortest path between them.

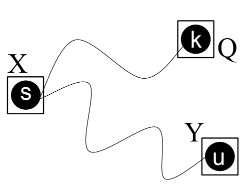


Fig. 7. Simultaneous simplification of two extrema.

Proposition 4. *If $t(X,Q)$ and $t(X,Y)$ are the tuples formed, respectively, by the connected regions $\{X,Q\}$ and $\{X,Y\}$, then the least significant regional extrema Q and Y merge simultaneously with X iff $\sigma(X,Q) = \sigma(X,Y)$, $\ell(X,Q) = \ell(X,Q)$ and $f(k) = f(u)$.*

Proof:

From the tuples definition above, we have:

$$t(X,Q) = t(\ \sigma(X,Q), \ell(X,Q)\ ) \qquad (15)$$

$$t(X,Y) = t(\ \sigma(X,Y), \ell(X,Y)\ ), \qquad (16)$$

Based on the order R in (5), a simultaneous merging of Q and Y with another extremum, X, leads to:

$$t(X,Q) = t(X,Y), \qquad (17)$$

which obviously implies:

$$\sigma(X,Q) = \sigma(X,Y) \qquad (18)$$

$$\ell(X,Q) = \ell(X,Y) \qquad (19)$$

Now, by considering $s \in X$, $k \in Q$ and $u \in Y$, where X, Q and Y are regional minima (maxima), then by the Equations (4) and (18):

$$\frac{|\ f(s) - f(k)\ |}{\ell(X,Q)} = \frac{|\ f(s) - f(u)\ |}{\ell(X,Y)} \qquad (20)$$

Since the two regional extrema Q and Y merge with a more significant extremum X, then the following relations hold for regional minima (dually for regional maxima):

$$f(s) < f(k) \qquad (21)$$

$$f(s) < f(u) \qquad (22)$$

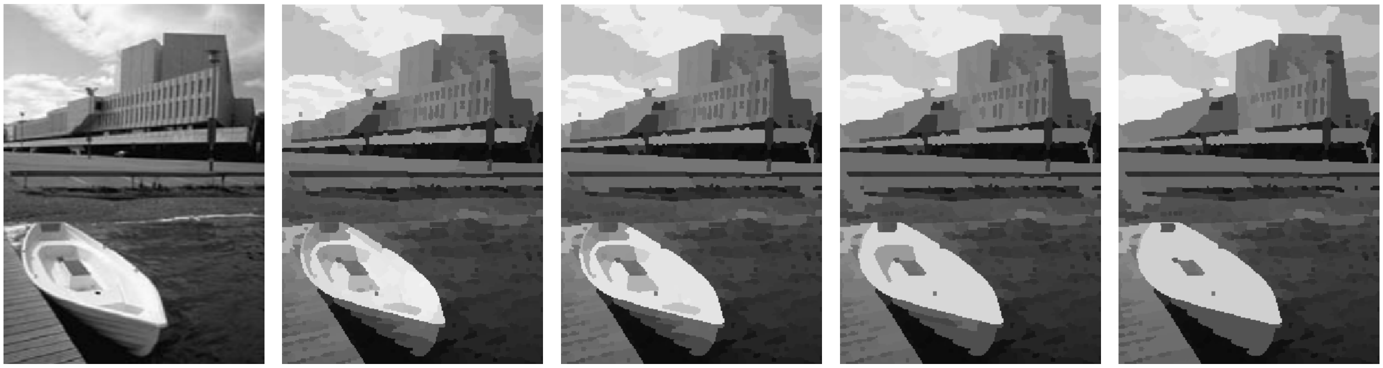
Finally, substituting Equations (21) and (22) into (20):

$$\begin{aligned}
\frac{f(s) - f(k)}{\ell(X,Q)} &= \frac{f(s) - f(u)}{\ell(X,Y)} \\
f(k) - f(s) &= f(u) - f(s) \\
f(k) &= f(u) \qquad (23)
\end{aligned}$$

■

The local update of graph G locally connecting the set of $s\text{-}neighbors$ regional extrema occurs as soon as a certain extremum merges with one its more relevant neighbors. This merging also implies the update of the remaining tuples concerned with the new configuration of the neighborhood. The following example illustrates how to approach this update. Let Q, X, Y and M be four regional minima (maxima) of a signal f. Also, let $s \in X$, $u \in Y$, $v \in Q$ and $k \in M$ be points of the corresponding extrema, as before. Fig. 9 gives an example of these points where Y is supposed to merge with one of its more significant neighbors.

If we consider, in this example, that X is the most significant neighbor of Y, followed by the regional extremum M, then the new scale, relating X with M, after the merging, is given



(a) Image original
min = 814 max = 824

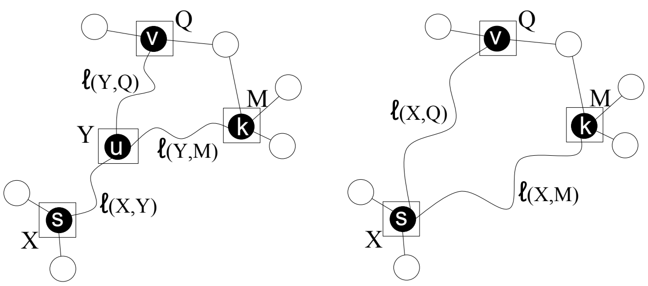
(b) $\sigma = 59.0$
min = 718 max = 630

(c) $\sigma = 10.9$
min = 610 max = 492

(d) $\sigma = 6.4$
min = 560 max = 385

(e) $\sigma = 5.1$
min = 407 max = 217

Fig. 8. Successive simplifications obtained from morphological reconstruction of the original image, using the extrema defined at each scale as markers. min and max indicates, respectivelly, the number of reamining regional minima and maxima at each iteration.



(a) Neighborhood configuration.

(b) Final graph update after suppression of vertice Y.

Fig. 9. Relationship update.

by the following function (according to Equation (4) and Propositions 2, 3 and 4).

$$F(X, Y, M) = \frac{|f(s) - f(k)|}{\ell(X, Y) + \ell(Y, M) + 1} \tag{24}$$

From Proposition 3, we also have

$$\ell(X, M) = \ell(X, Y) + \ell(Y, M) + 1, \tag{25}$$

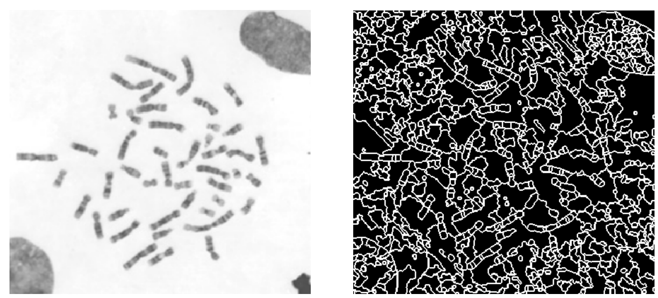
and the new tuple corresponds to $t_n = (F(X, Y, M), \ell(X, M))$. The new relation between vertices X and Q is obtained in the same way.

IV. Results and discussions

In this section, we compare the results of the simplification method obtained through the new properties detailed in Section III, and give some examples of the use of the operator τ (Equations 8 and 10) combined with different morphological tools. In these examples, a regular 8-connected grid graph is considered and the function z is given by the well-known watershed transform [16], [4].

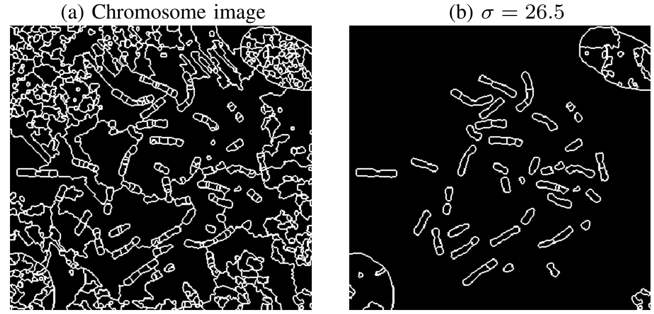
Fig. 8 shows a set of simplified images obtained from morphological reconstructions [2]. In such a case, the marker functions are represented by the input regional minima simplified with the tuples disposed in descending order. Note the significant reduction of the number of image extrema with level and position preservation of the remaining contours.

In Fig. 10 we illustrate how the watershed transform [16] can be combined with the simplification method to approach its typical over-segmentation problem. In these both cases, we use the regional minina obtained at each iteration as markers for the watershed function. Note the monotonic reduction of the over-segmented regions and the convergence of the method to a more meaningful and treatable segmentation result.



(a) Chromosome image

(b) $\sigma = 26.5$



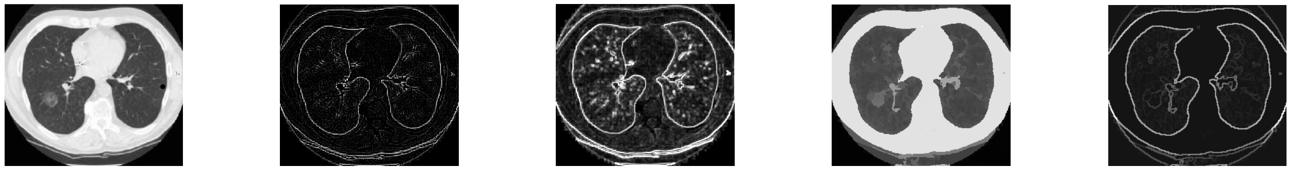
(c) $\sigma = 5.11$

(d) $\sigma = 0.9$

Fig. 10. Example of a simplification combined with the watershed transform.

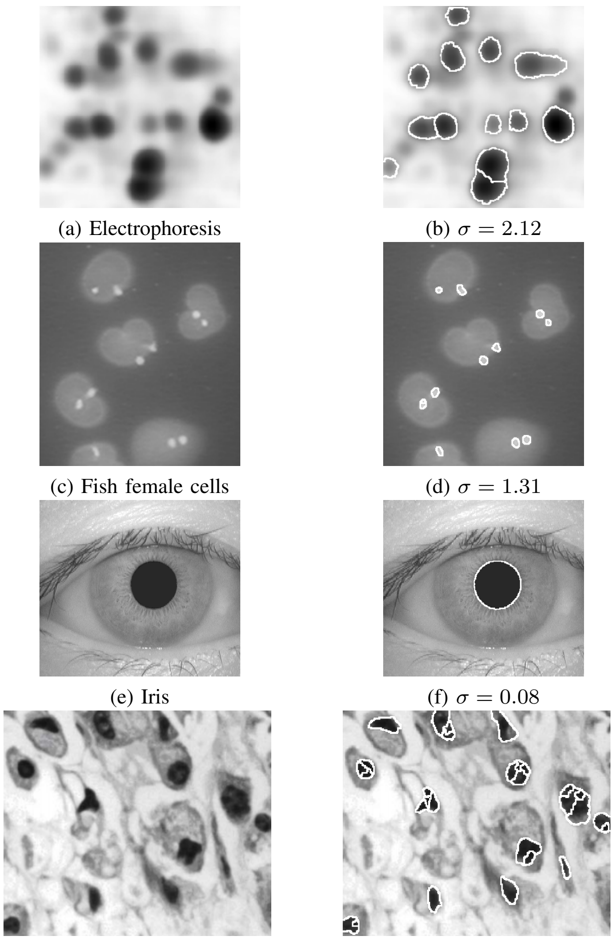
Fig. 11 shows the effect led by the homogenization of the regions when merged in a well-behaved manner. Fig. 11e depicts the contours of the homogenized image (Fig. 11d), highlighting its better quality w.r.t methods such as Laplacian and high-boost [17]. Finally, Fig. 12 gives some different examples of segmentation taking into account the simplification method explored in this work.

The improvement of the computational time achieved by considering the new properties described in Section III is



(a) Original (b) Laplacian contours (c) High boost contours (d) Homogenization (e) Homogenized contours

Fig. 11. Image homogenization example.



(a) Electrophoresis (b) $\sigma = 2.12$ (c) Fish female cells (d) $\sigma = 1.31$ (e) Iris (f) $\sigma = 0.08$ (g) Signet ring cancer cells (h) $\sigma = 0.77$

Fig. 12. Other segmentation examples.

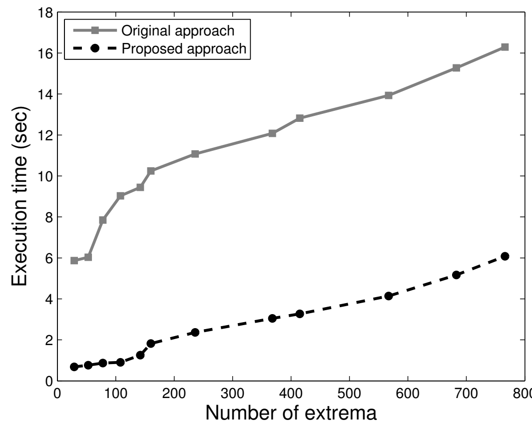


Fig. 13. Computational time comparison.

illustrated in Fig. 13, where the number of generated images is proportional to the number of image extrema. All the experiments were conducted using a system equipped with Intel Core i7-3770K processors per node, running at 3.50 Ghz with RAM 31Gb. In this case, the test image is of size 358×481 pixels (Fig. 8a) and the varying number of extrema was obtained by a simple blurring process.

V. Conclusion

In this paper, we dealt with a morphological simplification taking into account the information of distance (separation) and contrast between the extrema of a signal. We explored scale-space properties, such as monotonicity and continuity, as well as the graph-based implementation, to introduce new results concerned with the local update of a graph providing structure to a given image. These results led to significant improvements in terms of computational cost. Unlike the work in [12], which discussed mainly theoretical aspects of the original approach, we showed through some examples how to combine the simplification step with well-known morphological tools in applications related to image segmentation and homogenization.

Future works on this matter include, for example, the automatic and adaptive learning of the considered tuples, aiming at less supervised multi-scale tasks, and the investigation of the use of these tuples in problems related to image description and representation. Studies with different shapes of the structuring functions and other scaled operations such as erosion and dilations should also be of interest.

Acknowledgment

We gratefully acknowledge the partial financial support from CNPq-Brazil through grant 132866/2015-3.

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