# Automating the Generation of Procedures to Analyse Binary Images

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**Abstract.** A suitable model to represent procedures to analyse binary images is in terms of set transformations. Mathematical morphology is a general framework to represent set transformations as chains of four elementary operators (namely *erosion, anti-erosion* and their duals *dilation* and *anti-dilation*). In the present article, we present the general model proposed within the $\mathcal{AMM}$ project for automated generation of mathematical morphology procedures to analyse binary images. We also discuss one of the three identified strategies for simplification of the automatically generated procedures and show an example of its application.

## 1 Introduction

The analysis of binary (i.e. "black-and-white") images is an important tool in several areas as diverse as quantitative microscopy, cartography, office automation and industrial process control. Mathematical morphology, introduced in the 1960's by G. Matheron, J. Serra and colleagues at "École des Mines de Paris", is a general framework for set transformations which is particularly suitable to express translation-invariant procedures for image analysis [14, 16, 17]. Central to mathematical morphology is the decomposition of set transformations in terms of four elementary operators: dilation ($\delta$), erosion ($\epsilon$), anti-dilation ($\delta^a$) and anti-erosion ($\epsilon^a$). These operators admit *procedural interpretations* as abstract machines. To date, many implementations of the so-called *morphological machines* are available in hardware [12, 18, 7, 11, 6] and in software [13, 3, 5, 2].

We explore the problem of automatically generating *efficient* implementations of mathematical morphology procedures, and analyse an example of the particular problem of binary shape recognition. We develop our analysis based on the general model proposed within the $\mathcal{AMM}$ project, in which we identify three different strategies that can cooperate to generate efficient procedures. In the present article we explore one of them.

In order to make the paper self-contained, in section 2 we present the basic definitions of mathematical morphology and the formulation of the problem of shape recognition in binary images within this framework. In section 3 we briefly review the $\mathcal{AMM}$ model and discuss in some detail the *context-based simplification* strategy for mathematical morphology procedures. In section 4 we present an example of procedure generation and context-based simplification. Finally, in section 5 we present some further discussion and propose future research.

## 2 Mathematical Morphology

A binary image can be represented as subsets of a set $E \subseteq \mathbb{Z}^2$. An image transformation can thus be represented as a mapping from a collection of subsets of $E$ (the *input image*) to another (the *output image*).

Mathematical morphology is a framework to characterise translation-invariant set transformations. Let $2^E$ be the collection of all subsets of a finite non-empty set $E$. The set $E$ is assumed to be an abelian group with respect to a binary operation denoted by "+". The zero element of $(E, +)$ is denoted by 0.

For any $h \in E, X \subseteq E$, the set $X_h = \{y \in E.y = x + h, x \in X\}$ is called the *translate of* $X$ *by* $h$. In particular, $X_0 = X$.

A set transformation $\psi : 2^E \rightarrow 2^E$ is called *translation-invariant* iff $\psi(X_h) = \psi(X)_h, X \subseteq E, h \in E$. The *kernel* $\mathcal{K}(\psi)$ of a translation-invariant transformation $\psi$ is the subcollection of $2^E$ defined by $\mathcal{K}(\psi) = \{X \subseteq E.0 \in \psi(X)\}$.

Four translation-invariant transformations deserve special attention:

- a transformation $\epsilon_B$ is called an *erosion* if $\epsilon_B(X) = \{x \in E.B_x \subseteq X\}$;
- a transformation $\delta_B$ is called a *dilation* if $\delta_B(X) = \{x \in E B_x \cap X \neq \{\}\}$;
- a transformation $\epsilon^a_B$ is called an *anti-erosion* if $\epsilon^a_B(X) = \{x \in E B_x \cap X^c \neq \{\}\}$;
- a transformation $\delta^a_B$ is called an *anti-dilation* if $\delta^a_B(X) = \{x \in E.B_x \subseteq X^c\}$.

In all these transformations, $X^c$ denotes the complement to the set $X$ (i.e. $X^c = E \setminus X$), and the set $B \subseteq E$ is named the *structuring element* of the transformations.

Given two subsets $A, B \in 2^E, A \subseteq B$, a *closed interval* $[A, B]$ in $2^E$ is a collection of subsets of $E$ such that $[A, B] = \{X \subseteq E.A \subseteq X \subseteq B\}$. The subsets $A$ and $B$ are called respectively the *left* and *right extremities* of the closed interval.

A useful property of erosions and anti-dilations is that they can represent any translation-invariant transformation, as proved in [1] in the following

**Standard Decomposition Theorem:** Let $\psi$ be a translation-invariant transformation and $\mathcal{K}(\psi)$ be the kernel of this transformation. Then,

$$\psi(X) = \cup\{\epsilon_A(X) \cap \delta^a_{B^c}(X).[A, B] \subseteq \mathcal{K}(\psi)\}, X \subseteq E.$$

A dual result can be expressed in terms of dilations and anti-erosions.

The set $B(\psi)$ of all maximally closed intervals contained in $\mathcal{K}(\psi)$ is called the *basis* of $\psi$ (a closed interval contained in $\mathcal{K}(\psi)$ is *maximally closed* if it is not properly contained in any other interval contained in $\mathcal{K}(\psi)$).

It was also proved in [1] that $\mathcal{K}(\psi)$ can be replaced by $B(\psi)$ in the formulation of the Standard Decomposition Theorem, i.e. that

$$\psi(X) = \cup\{\epsilon_A(X) \cap \delta^a_{B^c}(X).[A, B] \subseteq B(\psi)\}, X \subseteq E.$$

In most practical situations, the transformations of interest are those that depend on a local neighbourhood of points only. A translation-invariant transformation is called *limited* iff there exists a window $W \subset E$ such that $h \in \psi(X) \leftrightarrow h \in \psi(X \cap W_h)$.

If $\psi$ is limited and $[A, B] \in B(\psi)$, then $A, B^c \subseteq W$. In other words, the structuring elements of the erosions and anti-dilations of $\psi$ are subsets of $W$.

An important feature of translation-invariant transformations is that they are closely related to boolean functions: for each limited translation-invariant transformation $\psi$ there exists a corresponding boolean function $b : 2^W \rightarrow \{\perp, \top\}$ such that $b(X) = \top \leftrightarrow 0 \in \psi(X)$, and for each boolean expression $b$ constructed from $2^W$ there exists a corresponding limited translation-invariant transformation given by $\psi(X) = \{h \in E.b(X \cap W_h) = \top\}$. Once a boolean expression $b(X)$ as above is transformed to minimal disjunctive normal form, the positive and the negated variables correspond respectively to the left and to the complement of the right extremities of the basis of the corresponding transformation $\psi(X)$. This property was explored in [4] to obtain optimal morphological filters for image analysis.

A transformation $\psi$ is called a ***window transformation with respect to*** $W \subset E$ iff there exists a subcollection $D \subseteq 2^W$ such that $\psi(X) = \{h \in E.W \cap X_{-h} \in D\}$. A window transformation can be used to "tag" particular shapes in $D$ by producing point markers, thus working as a shape recognition procedure. Window transformations are translation-invariant, and the kernel of a window transformation $\psi$ is given by $\mathcal{K}(\psi) = \{X \subseteq E.W \cap X \in D\}$.

## 3 The $\mathcal{AMM}$ Model

The general model for automating the generation of efficient procedures for mathematical morphology is presented in figure 1. In this model, three different approaches cooperate to generate mathematical morphology procedures: transformation of propositional expressions representing classes of images, PAC-learning of the same classes of images, and direct procedure derivation in terms of high-level, abstract descriptions of image transformations using theorem proving techniques.

Images and image transformations can be envisaged as propositional formulae in which basic propositions represent points in $\mathbb{Z}^2$. In order to make the correspondence between images and propositions more natural, we adopt a non-conventional way of indexing basic propositions (see figure 2). The alphabet of our propositional language is defined as follows:

- a countable set of basic propositions
  $$\Phi = \{ p_{00}, q_{00}, p_{10}, q_{10}, p_{11}, q_{11}, p_{01}, q_{01}, p_{-11}, q_{-11}, p_{-10}, q_{-10}, p_{-1-1}, q_{-1-1}, p_{0-1}, q_{0-1}, p_{1-1}, q_{1-1}, \ldots\}.$$
- the conventional connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Intuitively, propositions $p_{ij}$ represent points of "input" images and propositions $q_{ij}$ represent points of "output" (i.e. "transformed") images. To "true" basic propositions correspond "black" points, and to "false" basic propositions correspond "white" points. Finally, an image transformation is represented as a collection of formulae of the form $P \leftrightarrow Q$, in which $P$ contains only propositions $p_{ij}$ and $Q$ contains only propositions $q_{ij}$.

A translation-invariant transformation $\psi$ can be represented as a single formula $P_\psi \leftrightarrow q_{00}$, in which $P_\psi$ represents the local rule which characterises the transformation. Given a translation-invariant transformation $\psi$, we can construct the boolean expression corresponding to the basis $B(\psi)$ of $\psi$, which characterises a procedural description of $\psi$: assuming $P_\psi$ as being in disjunctive normal form, the basis

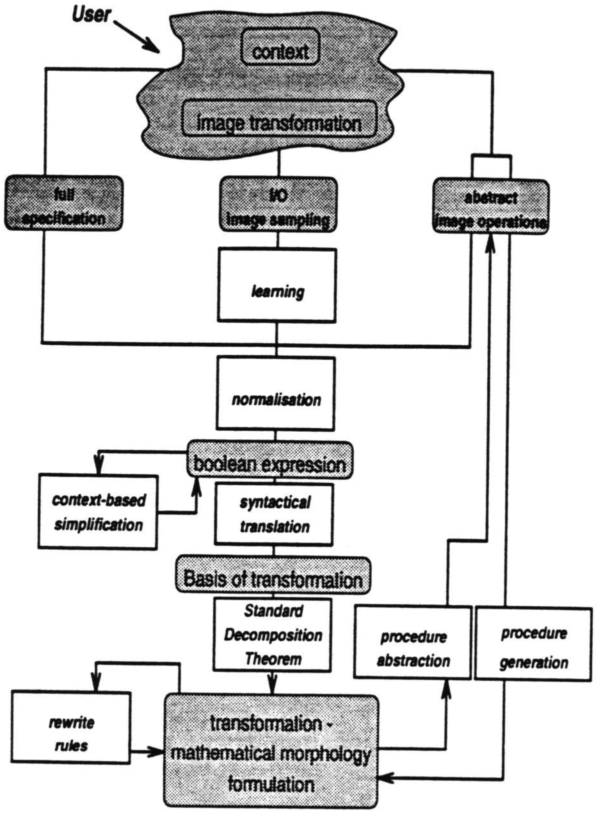


Figure 1: General $\mathcal{AMM}$ Model

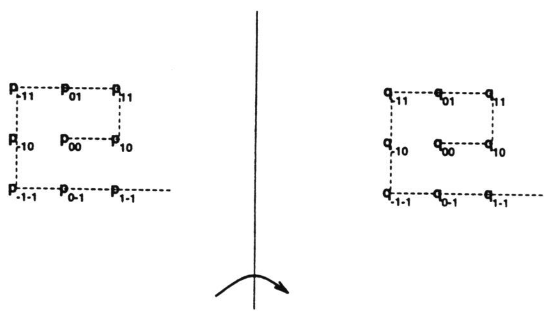

Figure 2: Correspondence between Basic Propositions and Points

$B(\psi)$ is given by the intervals $[A_i, B_i^c]$ where $A_i$ and $B_i$ are respectively the points corresponding to the "positive" and to the "negated" basic propositions ocurring in each disjunct $\varphi_i \in P_\psi$. Thus, given a complete but informal specification of the $\psi$, once it is rewritten as a propositional formula $P_\psi \leftrightarrow q_{00}$ and then *normalised* to disjunctive normal form, we obtain the "canonical implementation" of $\psi$ in terms of its basis.

In many practical situations, however, we cannot count on a complete specification of $\psi$. Indeed, what we do have most frequently for a transformation $\psi$ is a (non-exhaustive) collection of "input/output" examples. In these cases, the best we can do is approximate the specification of $\psi$ via *inductive learning* of the corresponding boolean function, e.g. using a PAC-learning algorithm [19, 10].

We frequently have additional knowledge about the desired transformation in terms of *abstractions* of "chunks" of mathematical morphology basic operators that yield to specific effects on generic images – e.g. border extraction, skeletisation, etc. – which are used to guide (and sometimes to specify completely) the implementation of image transformations. In such cases, we can use automatic theorem proving techniques to synthesise the corresponding morphological operators [9].

We consider three independent ways of simplifying the implementation of $\psi$ corresponding to $B(\psi)$:

1. via *context-based simplification*;
2. via the appropriate utilisation of algebraic *rewrite rules* that preserve the specification of $\psi$ [21, 8]; and
3. via the appropriate simplification of the subprocedures ocurring in $\psi$ based on their *abstract interpretations* as "chunks" of operators for image transformations rather than general propositional formulae, in a similar vein to the "automatic" generation of mathematical morphology procedures suggested in [9].

In the following paragraphs we discuss the first of these simplification strategies (which is the simplest of the three).

By *context-based simplification* we understand the appropriate use of available *a priori* information about the family of "input" images. This information is encoded as a disjunction of propositional formulae $\hat{P}_i$ corresponding to the shapes that actually occur as "inputs". In other words, if we have *a priori* information about the collection of shapes that we can get as "inputs"(i.e. our *context*), we can reduce

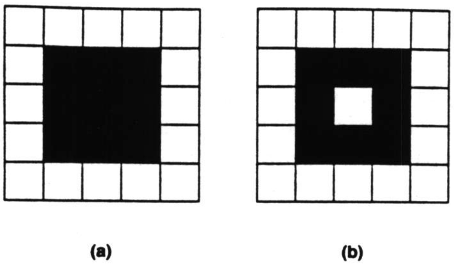


Figure 3: A $3 \times 3$ Isolated Square and a $3 \times 3$ Square with a Hole in the Middle

our shape recognition task to the task of identifying only the distinguishing features among these shapes.

Given *a priori* information that our context is bound to the set $C = \{\hat{P}_1, ..., \hat{P}_n\}$, a shape recognition transformation $\psi = P_\psi \leftrightarrow q_{00}$ can be restated as $\hat{P}_\psi \leftrightarrow q_{00}$, where $\hat{P}_\psi$ takes into account that any occurring shape must be an element of $C$. Usually (but not necessarily) this information simplifies the "canonical implementation" of $\psi$, as the example in the following section illustrates. Similarly to the shapes to be recognised, context information may also be given via examples and approximated via inductive learning.

## 4 An Example

Let the shape to be recognised be a $3 \times 3$ isolated square, i.e. a square with no holes and sizes 3 points long, in which all neighbouring points are "black" (figure 3(a)). This simple shape recognition task can be characterised as:

$$\bigwedge_{-1\leq i,j\leq 1} p_{ij} \bigwedge_{-2\leq i\leq 2} \neg p_{i2} \bigwedge_{-2\leq i\leq 2} \neg p_{i-2} \bigwedge_{-1\leq j\leq 1} \neg p_{2j} \bigwedge_{-1\leq j\leq 1} \neg p_{-2j} \leftrightarrow q_{00}.$$

The corresponding basis of the transformation is given by the single interval $[A, B]$, where $A = \{-1, 0, 1\}^2$ and $B^c = \{-2, -1, 0, 1, 2\}^2 \setminus A$ (figure 4(a)).

Now assume that we know *a priori* that the only occurring shapes in our "input" images are isolated squares with no holes (figure 3(a)) and isolated squares with a hole in the middle (figure 3(b)) - all $3 \times 3$ squares. In other words, we know *a priori* that whenever we find a $3 \times 3$ "cross", this "cross" must belong to the target shape. The simplification of $\psi$ is obtained as follows:

- $C = \{\hat{P}_1, \hat{P}_2\}$
- $\hat{P} =$
$\bigwedge_{-1\leq i,j\leq 1} p_{ij} \bigwedge_{-2\leq i\leq 2} \neg p_{i2} \bigwedge_{-2\leq i\leq 2} \neg p_{i-2} \bigwedge_{-1\leq j\leq 1} \neg p_{2j} \bigwedge_{-1\leq j\leq 1} \neg p_{-2j}.$

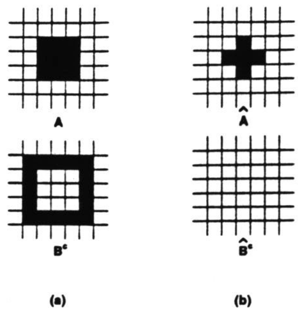


Figure 4: Bases $[A, B]$ and $[\hat{A}, \hat{B}]$

- $\hat{P}_2 =$
$\bigwedge_{i,j\in\{-1,1\}} p_{ij} \bigwedge_{i\in\{-1,1\}} p_{i0} \bigwedge_{j\in\{-1,1\}} p_{0j} \wedge \neg p_{00} \bigwedge_{-2\leq i\leq 2} \neg p_{i2} \bigwedge_{-2\leq i\leq 2} \neg p_{i-2} \bigwedge_{-1\leq j\leq 1} \neg p_{2j} \bigwedge_{-1\leq j\leq 1} \neg p_{-2j}.$
- $[(\bigwedge_{i\in\{-1,1\}} p_{i0} \bigwedge_{j\in\{-1,1\}} p_{0j}) \leftrightarrow \hat{P}_1 \vee \hat{P}_2] \wedge [\hat{P}_1 \leftrightarrow q_{00}]$.

The normalisation of this last expression gives:

$$\bigwedge_{i\in\{-1,1\}} p_{i0} \bigwedge_{j\in\{-1,1\}} p_{0j} \wedge p_{00} \leftrightarrow q_{00}.$$

The basis of this transformation can now be given by the interval $[\hat{A}, \hat{B}]$, where $\hat{A} = \{(-1, 0), (1, 0), (0, 0), (0, -1), (0, 1)\}$ and $B^c = \{\}$ - which is clearly "simpler" than $[A, B]$, as it requires less points to be constructed (figure 4(b)).

## 5 Discussion and Future Work

In this paper we briefly reviewed the general model proposed in the $\mathcal{AMM}$ project to automatically generate implementations of mathematical morphology operators, and discussed one of the three simplification strategies considered in this project. Assuming a complete specification of shapes occurring in input images and of the shape(s) to be recognised, we presented a general procedure that can generate the corresponding image transformation $\psi$. Furthermore, we showed how *a priori* information can be used to simplify the implementation of this transformation.

The application of artificial intelligence techniques to automate the generation of procedures for image analysis has been studied by other authors (e.g. [15, 20, 22]. To our knowledge, however, we are the first ones to explore the integration of an extensive use of the result conveyed by the Standard Decomposition Theorem with PAC-learning techniques and automatic theorem proving techniques to synthesise mathematical morphology operators, together with algebraic rewrite rules, heuristic procedure abstraction and context-based simplification to improve the efficiency of these operators.

Immediate future work in the $\mathcal{AMM}$ project includes:

- the analysis of patterns of input images to characterise in which cases the utilisation of *a priori* information can be profitable;
- a more thorough analysis of the algebraic properties of the basic operators of mathematical morphology, so that we can specify and implement a rewrite system to simplify chains of operators;
- the abstract analysis based on the *intended meanings* of specific chains of operators, and the construction of a library of subprocedure simplifications based on such meanings;
- the analysis of the computational complexity of learning shapes to be recognised and classes of *a priori* occurring shapes.

All these future research goals shall be implemented to constitute an integrated environment for the automatic generation of mathematical morphology procedures.

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## References

[1] G. J. F. Banon and J. Barrera. Minimal Representations for Translation-invariant Set mappings by Mathematical Morphology. *SIAM Journal of Applied Mathematics*, 51:1782–1798, 1991.

[2] J. Barrera, G. J. F. Banon, and R. A. Lotufo. A Mathematical Morphology Toolbox for the KHOROS System. 1993.

[3] S. Beucher. MICROMORPH - Manuel de Reference. Technical report, Ecole des Mines de Paris, 1987.

[4] E. R. Dougherty. Optimal Mean-square n-Observation Digital Morphological Filters. *CVGIP: Image Understanding*, 55:36–54, 1992.

[5] C. Gritin. *De la Representation des Images au Traitement Morphologique d'Images Tridimensionnelles*. PhD thesis, Ecole des Mines de Paris, 1993.

[6] K. S. Huang, B. K. Jenkins, and A. A. Sawchulk. Binary Image Algebra and Optical Cellular Logic Processor Design. *Computer Vision, Graphics and Image Processing*, 45:295–345, 1989.

[7] P. Husson, J. P. Derutin, P. Bonton, and J. Gallice. Elementary Processor with Mathematical Morphology Implemented in VLSI and Intended for Systolic Architecture. In *Signal Processing IV: Theories and Application*, 1988.

[8] R. Jones. Transformation of the Basis Representation into Cascade Representations. In *International Workshop on Mathematical Morphology and its Applications to Signal Processing*, 1993.

[9] H. Joo, R. M. Haralick, and L. G. Shapiro. Toward the Automatic Generation of Mathematical Morphology Procedures Using Predicate Logic.

[10] L. P. Kaelbling. *Learning in Embedded Systems*. PhD thesis, Stanford University, 1990.

[11] J. C. Klein and R. Peyhard. PIMM1, an Image Processing ASIC Based on Mathematical Morphology. In *IEEE ASIC Seminar and Exhibit*, 1989.

[12] J. C. Klein and J. Serra. The Texture Analyser. *Journal of Microscopy*, 95:343–356, 1972.

[13] B. Lay. Description des Programmes du Logiciel Morphologique. Technical Report N-904, CGMM - Fontainebleau, 1984.

[14] G. Matheron. *Random Sets and Integral Geometry*. John Wiley, 1975.

[15] M. Schmitt. *Des Algorithmes Morphologiques a l'Intelligence Artificielle*. PhD thesis, Ecole des Mines de Paris, 1989.

[16] J. Serra. *Image Analysis and Mathematical Morphology – v. 1*. Academic Press, 1982.

[17] J. Serra. *Image Analysis and Mathematical Morphology – v. 2*. Academic Press, 1988.

[18] S. R. Sternberg. Cellular Computers and Biomedical Image Processing. In J. Skalansky and J. C. Bisconte, editors, *Biomedical Images and Computers*. Springer Verlag LNMI, 1982.

[19] L. G. Valiant. A Theory of the Learnable. *Comm. ACM*, 27:1134–1142, 1984.

[20] R. C. Vogt. *Automatic Generation of Morphological Set Recognition Algorithms*. Springer-Verlag, 1991.

[21] J. Xu. Decomposition of Convex Polygonal Morphological Structuring Elements into Neighbourhood Subsets. *IEEE Transactions PAMI*, 13:153–162, 1993.

[22] H Zhao. *Systeme Expert et Morphologie Mathematique*. PhD thesis, Ecole des Mines de Paris, 1992.