Geodesic conic subdivision curves on surfaces

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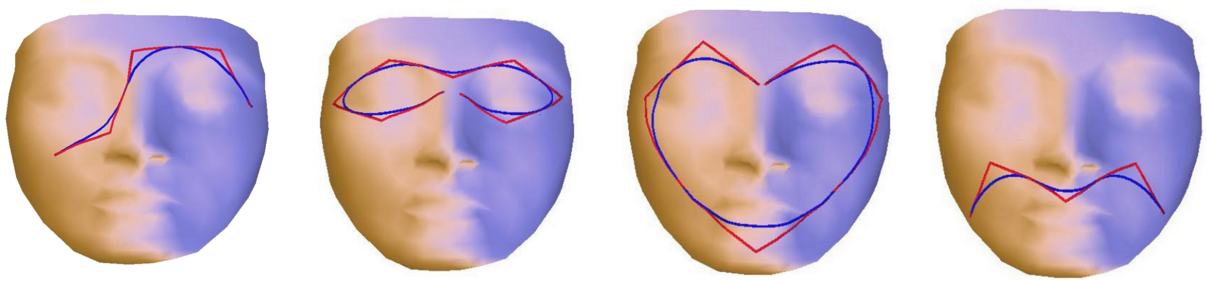

Fig. 1. Geodesic conic subdivision curves on a triangulated surface. The initial control polygon (in red) and the curve (in blue).

***Abstract*—In this paper we present a nonlinear curve subdivision scheme, suitable for designing curves on surfaces. The scheme is inspired in the concept of geodesic Bézier curves. Starting with a geodesic control polygon with vertices on a surface S, the scheme generates a sequence of geodesic polygons that converges to a continuous curve on S. If the surface is C^2 continuous, then the subdivision curve is C^1 continuous. In the planar case, the limit curve is a conic Bézier spline curve. Each section of the subdivision curve, corresponding to three consecutive points of the control polygon, depends on a free parameter which can be used to obtain a local control of the shape of the curve. Results are extended to triangulated surfaces showing that the scheme is suitable for designing curves on these surfaces and has the convex hull property.**

***Keywords*-Geodesic; conics; subdivision curves**

I. Introduction

A. Motivation

Designing free-form curves is a basic operation in Geometric Modeling. In the Euclidean space it is a widely studied problem, nevertheless it becomes much harder if we wish to design on a curved geometry, such as a triangulated surface. The problem has been addressed on smooth manifolds as well as on triangulations, see for instance [1], [2], [8].

Subdivision methods are currently very popular as a design tool, since subdivision curves can be easily computed in the Euclidean space. Nevertheless, their counterpart on curved surfaces are more involved and expensive. A first step on this sense are linear subdivision schemes on smooth and discrete manifolds [9], [10], [12], [15]. Nonlinear schemes, which arise as perturbations of linear schemes on smooth manifolds, are the next step. They have been described by Wallner and Pottmann in [20]. Several examples where nonlinear subdivision schemes are useful in Computer Graphics are also presented in [20]. The convergence and smoothness analysis of these subdivision schemes can be found in the work of Wallner and Dyn [19]. They generalize the linear schemes to manifolds in two different ways. The first approach substitutes linear average by geodesic average. This method is very good because it is completely intrinsic, although for some schemes it requires to compute many geodesics. The second method performs each subdivision step in the ambient space, projecting the new points into the manifold. This approach is more efficient, but depending on the complexity of the geometry it could conduce to wrong or unexpected results. Some variants of de Casteljau's Algorithm have been also used to define curves on Riemannian manifolds [17] and Lie groups [2].

In [13] an algorithm to compute a geodesic path over a triangulated surface is presented. This algorithm is used to define geodesic Bézier curves [14]. They are a natural extension of Bézier curves in the sense that linear interpolation is substituted by geodesic interpolation. In [15] a simple method to define subdivision schemes on triangulations is proposed. Using both, shortest and straightest geodesics, a perturbation of a planar binary subdivision is translated on the triangulation. This method allows to extend to a triangulated surface any binary subdivision scheme, regardless whether it is linear or not.

Inspired in these ideas we introduce in the present paper a natural extension of geodesic Bézier curves [14] for the rational quadratic case: geodesic conic Bézier curves. They are defined as subdivision curves on a surface. More precisely, starting with a set of points on a surface S, a control polygon composed by geodesic arcs joining two consecutive points is defined. In each step a new geodesic polygon is computed defining a subdivision scheme that converges to a continuous curve living on S.

In the planar case the subdivision curve is a conic Bézier spline curve. Results are extended to triangulated surfaces showing that the scheme is suitable for designing curves on these surfaces and may be useful for trimming and segmentation, see Figure 1.

B. Our contribution

The main contribution of this paper is the definition of geodesic conic curves as the limit of a carefully designed subdivision scheme. This scheme is obtained from a natural generalization of de Casteljau rational algorithm, where linear interpolation is substituted by geodesic interpolation and the subdivision parameter is chosen in such away that the left and right segments have the same weighs in the standard representation. The resulting scheme is based in the shoulder point and it is very simple. Its implementation is straightforward if an efficient procedure for computing geodesic curves is available.We also provide a rigorous analysis of the convergence and smoothness of the scheme, showing that under mild conditions the geodesic conic subdivision curve is C^1 continuous if the surface is C^2.

In comparison with other subdivision schemes for generating curves on surfaces, the proposed scheme has very nice properties. It can be used to design curves on surfaces with relative complex topology, having eventually very close branches (without self-intersections) and with a finite number of holes. Moreover, the scheme has free parameters that are very useful to control the shape of the subdivision curve. Finally, it enjoys the convex hull property and in consequence the subdivision curve has not undesirable oscillations or inflection points.

The rest of the paper is organized as follows. In section 2 we introduce the notation and the classical planar subdivision scheme for conics. In section 3 we define the geodesic conic subdivision scheme on surfaces and analyze its convergence and smoothness. Section 4 is devoted to geodesic conic curves on triangulated surfaces. We include in this section details of the user interface and several examples. Finally, in section 5 we give concluding remarks.

II. BASIC THEORY: THE SUBDIVISION SCHEME FOR CONICS

A rational Bézier curve of degree n is a parametric curve which is described by $n+1$ control points, $b_i \in R^m,\ m = 2, 3$ and $n+1$ weights ω_i. For $t \in [0, 1]$ the curve has the form

$$c(t) = \frac{\sum_{i=0}^{n} \omega_i b_i B_i^n(t)}{\sum_{i=0}^{n} \omega_i B_i^n(t)}$$

where $B_i^n(t),\ i = 0, 1, ..., n$ are the Bernstein Bézier basis functions of degree n, [7]. Conics are rational Bézier curves of degree $n = 2$. It has been shown [16] that without loss of generality we may assume that any nondegenerate conic may be written in *standard representation*, where $\omega_0 = \omega_2 = 1$. Since in what follows all Bézier conics are in standard representation, for the sake of simplicity we will not mention explicitly the whole set of homogeneous weights $\omega_i,\ i = 0, 1, 2$ and we will denote the weight $\omega_1 > 0$ by $\omega > 0$. The intersection point between the segment of $c(t)$ inside the triangle with vertices b_0, b_1, b_2 and the line passing through b_1 and $\frac{b_0+b_2}{2}$ is called *shoulder point* [7]. If $c(t)$ is in the standard representation then the shoulder point s is $c(\frac{1}{2})$.

Rational Bézier curves may be evaluated by de Casteljau algorithm [5]. In the case of conics this algorithm is described as follows.

procedure DECASTELJAU($b_0, b_1, b_2, \omega_0, \omega_1, \omega_2, t$)
  for $i = 0, 1, 2$ **do**
    $b_i^0(t) = b_i,\ \ \omega_i^0(t) = \omega_i$
  end for
  for $j = 1, 2$ **do**
    for $i = 0, ..., 2 - j$ **do**
      $\omega_i^j(t) = (1-t)\omega_i^{j-1}(t) + t\omega_{i+1}^{j-1}(t)$
      $b_i^j(t) = (1-t)\frac{\omega_i^{j-1}(t)}{\omega_i^j(t)}\, b_i^{j-1}(t) + t\frac{\omega_{i+1}^{j-1}(t)}{\omega_i^j(t)}\, b_{i+1}^{j-1}(t)$
    end for
  end for
  return $c(t) = b_0^2(t)$ ▷ Point for the parameter value t
end procedure

The intermediate Bézier points $b_i^j(t)$ of the above algorithm may be used to subdivide the curve c at parameter value $t \in (0, 1)$. More precisely, the left segment of c corresponding to the parameter values in the interval $[0, t]$ is a quadratic rational Bézier curve $c_0^1(u),\ u \in [0, 1]$ with control polygon $b_0, b_0^1(t), b_0^2(t)$ and weights $1, \omega_0^1(t), \omega_0^2(t)$. Similarly, the right segment of c corresponding to the parameter values in $(t, 1)$ is a quadratic rational Bézier curve $c_1^1(u),\ u \in [0, 1]$ with control polygon $b_0^2(t), b_1^1(t), b_2$, and weights $\omega_0^2(t), \omega_1^1(t), 1$. Algorithm BASICCLASSICSUBD describes the basic subdivision.

procedure BASICCLASSICSUBD($b_0, b_1, b_2, \omega_0, \omega_1, \omega_2, t$)
  DECASTELJAU($b_0, b_1, b_2, \omega_0, \omega_1, \omega_2, t$)
  $P_0 \leftarrow [b_0, b_0^1(t), b_0^2(t)]$, $\Omega_0 \leftarrow [1, \omega_0^1(t), \omega_0^2(t)]$
  $P_1 \leftarrow [b_0^2(t), b_1^1(t), b_2]$, $\Omega_1 \leftarrow [\omega_0^2(t), \omega_1^1(t), 1]$
  return $\{P_0, \Omega_0, P_1, \Omega_1\}$
end procedure

This process may be repeated, subdividing each conic segment $c_0^1(u), c_1^1(u)$ in a parameter value $u \in (0, 1)$, for instance $u = \frac{1}{2}$. If we use this subdivision, after j steps we obtain 2^j control polygons (and the corresponding weights) that allow to represent a segment of the (unique) conic curve $c(t), t \in [0, 1]$ as a Bézier rational quadratic curve. When $j \to \infty$, this sequence of control polygons tends to the conic curve. In this paper, we will refer to this subdivision scheme, based on the dyadic parameters, as the *classic subdivision scheme*. Recall that even if we start with the standard representation of c, if we subdivide it in $t = \frac{1}{2}$ using the classic scheme, then $c_0^1(\frac{1}{2})$ is not necessarily neither the shoulder point of $c_0^1(u)$ nor the point $c(\frac{1}{4})$ (by the same reason $c_1^1(\frac{1}{2})$ is not necessarily neither the shoulder point of $c_1^1(u)$ nor the point $c(\frac{3}{4})$), see [5].

A different scheme, converging to the same curve, may be obtained if we make a standardization of the conics in each step. In fact, since the weight $\omega_0^2(t)$ in Algorithm BASICCLASSICSUBD is not necessarily equal to 1, to write the left and the right segment of the conic in the standard form we have to introduce the following substitutions [6],

$$\omega_0^1(t) \leftarrow \frac{\omega_0^1(t)}{\sqrt{\omega_0^2(t)}}, \;\; \omega_1^1(t) \leftarrow \frac{\omega_1^1(t)}{\sqrt{\omega_0^2(t)}}, \;\; \omega_0^2(t) \leftarrow 1 \tag{1}$$

For a rational Bézier conic in *standard representation* the Farin points q_0, q_1 are characterized by the fact that $\omega = ratio(b_i, q_i, b_{i+1}),\;\; i = 0, 1$. In terms of the control points $b_i,\; i = 0, 1, 2$, they can be expressed as

$$q_0 = \frac{b_0 + \omega b_1}{1+\omega}\;,\; q_1 = \frac{b_2 + \omega b_1}{1+\omega} \tag{2}$$

From Algorithm DECASTELJAU it is easy to check that $q_0 = b_0^1(\frac{1}{2})$ and $q_1 = b_1^1(\frac{1}{2})$. Moreover, $\omega_0^1 = \omega_1^1 = \omega_0^2 = \frac{1+\omega}{2}$ and after the standardization (1) we obtain,

$$\omega_0^1 = \omega_1^1 = \sqrt{\frac{1+\omega}{2}} \tag{3}$$

Hence, if we subdivide a rational Bézier conic curve c in the *standard representation* at the *shoulder point* $s = c(\frac{1}{2})$, then we obtain two arcs of the same conic that can be written in the standard Bézier representation. The left arc $\overline{c}_0^1(u), u \in [0,1]$ corresponding to the interval $t \in [0, \frac{1}{2}]$ has control points b_0, q_0, s and weights $1, \sqrt{\frac{1+\omega}{2}}, 1$, while the right arc $\overline{c}_1^1(u), u \in [0,1]$ corresponding to the interval $t \in [\frac{1}{2}, 1]$ has control points s, q_1, b_2, and weights $1, \sqrt{\frac{1+\omega}{2}}, 1$. Observe that the weighs of both segments are the same. Algorithm BASICSHOULDERPSUBD describe this subdivision step.

procedure BASICSHOULDERPSUBD(b_0, b_1, b_2, ω)
  $q_0 \leftarrow \frac{b_0+\omega b_1}{1+\omega}, q_1 \leftarrow \frac{b_2+\omega b_1}{1+\omega}, s \leftarrow \frac{q_0+q_1}{2}, \omega^1 \leftarrow \sqrt{\frac{1+\omega}{2}}$
  $\widetilde{P}_0 \leftarrow [b_0, q_0, s], \quad \widetilde{\Omega}_0 \leftarrow [1, \omega^1, 1]$
  $\widetilde{P}_1 \leftarrow [s, q_1, b_2], \quad \widetilde{\Omega}_1 \leftarrow [1, \omega^1, 1]$
  return $\{\widetilde{P}_0, \widetilde{\Omega}_0, \widetilde{P}_1, \widetilde{\Omega}_1\}$
end procedure

This process may be repeated, subdividing $\overline{c}_0^1(u)$ and $\overline{c}_1^1(u)$ in their shoulder points by means of the Algorithm BASICSHOULDERPSUBD. We call this scheme *Basic shoulder point subdivision scheme*. For $j \rightarrow \infty$, the sequence of control polygons obtained tends to the conic curve.

Summarizing, if we apply Algorithm BASICCLASSICSUBD with $t = \frac{1}{2}$ and Algorithm BASICSHOULDERPSUBD to the standard Bézier representation of a conic, we obtain the same control polygons but with *different weights*. Hence, if we repeat the process and subdivide in $u = \frac{1}{2}$ with Algorithm BASICCLASSICSUBD the control polygons of the segments $c_0^1(u)$ and $c_1^1(u)$ previously obtained, then the results are different from those obtained subdividing at the shoulder point with Algorithm BASICSHOULDERPSUBD the control polygon of the curves $\overline{c}_0^1(u)$ and $\overline{c}_1^1(u)$. In other words, the sequence of control polygons generated by the *classic subdivision scheme* and the *shoulder subdivision scheme* are different, as shown in Figure 2.

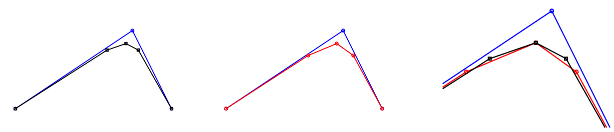

Fig. 2. Control polygon P^0 (blue) and the polygonal curves P^2 after two subdivision steps. Left:P^0 and the polygonal curve P^2 (black) obtained with the classic subdivision. Center:P^0 and the polygonal curve P^2 (red) obtained with the shoulder point scheme. Right: P^0 and both polygonal curves P^2, zoom of the central region.

Applying recursively the *shoulder point subdivision*, we obtain the following subdivision scheme that generates in the limit a *piecewise conic curve*.

Given a sequence of points on the plane

$$P^0 = \{P_0^0, P_1^0, P_2^0, ..., P_{2n-1}^0, P_{2n}^0\}$$

and a local tension parameter $\omega_i^0 > 0$ associated to the subsequence $P_i^0, P_{i+1}^0, P_{i+2}^0,\;\; i = 0, 2, ..., 2n-2$, the subdivision rule is based on the recurrences (2) and (3). More precisely, for the $P_i^0, P_{i+1}^0, P_{i+2}^0$, with i even and $w_i^0 > 0$, it is given by (see Figure 3),

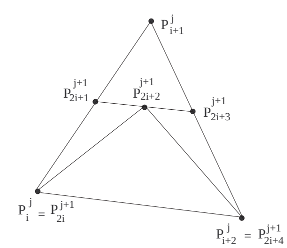

Fig. 3. Control polygons of two consecutive steps

Shoulder point conic subdivision

$$\begin{aligned}
P_{2i}^{j+1} &= P_i^j \qquad (4)\\
P_{2i+1}^{j+1} &= (1-\gamma_{2i}^{j+1})\, P_i^j + \gamma_{2i}^{j+1}\, P_{i+1}^j \qquad (5)\\
P_{2i+3}^{j+1} &= \gamma_{2i}^{j+1}\, P_{i+1}^j + (1-\gamma_{2i}^{j+1})\, P_{i+2}^j \qquad (6)\\
P_{2i+2}^{j+1} &= \frac{1}{2} P_{2i+1}^{j+1} + \frac{1}{2} P_{2i+3}^{j+1} \qquad (7)
\end{aligned}$$

where the tension parameters of the step $j+1$ are computed as follows,

$$\omega_{2i}^{j+1} = \omega_{2i+2}^{j+1} = \sqrt{\frac{1+\omega_i^j}{2}} \quad , \quad \gamma_{2i}^{j+1} = \gamma_{2i+2}^{j+1} = \frac{\omega_{2i}^{j+1}}{1+\omega_{2i}^{j+1}}$$

From the previous relations it is straightforward to obtain the following recursion

$$\gamma_{2i}^{j+1} = \frac{1}{1+\sqrt{2(1-\gamma_i^j)}} \tag{8}$$

starting with

$$\gamma_i^0 = \frac{\omega_i^0}{1+\omega_i^0} \tag{9}$$

Remarks
The points P_{2i+1}^{j+1} and P_{2i+3}^{j+1} play the role of the Farin points for the subsequence $P_i^j, P_{i+1}^j, P_{i+2}^j$. Moreover, if the points of the subsequences $P_{2i-1}^0, P_{2i}^0, P_{2i+1}^0, i = 1, ..., n-1$ are collinear, then the subdivision curve is a G^1-continuous conic Bézier spline. Observe that $\omega_i^0 > 0$ implies $0 < \gamma_i^0 < 1$. Futhermore, from (9) and (8) it is easy to check that $\lim_{j\to\infty} \gamma_m^j = \frac{1}{2}$.

III. The conic subdivision scheme on surfaces

In this section we introduce *geodesic conic curves on surfaces* as the limit of a subdivision scheme, which can be considered as a natural generalization of the *shoulder point conic subdivision scheme* (4)-(7). Observe that the shoulder point scheme is also well defined if the points of the initial polygon P^0 are in R^3. Nevertheless, if they are on a surface S and we apply directly the shoulder point subdivision, the new points P^1 are not necessarily on S. A way of solving this problem is substituting straight lines in affine space by geodesic lines on the surface.

A. Definition of the scheme

Assume that S is an smooth surface and Q_0, Q_1 two points in S. We denote by $c_g(Q_0, Q_1)$ the shortest geodesic curve with initial point Q_0 and final point Q_1 and denote by $d_g(Q_0, Q_1)$ the arc-length of $c_g(Q_0, Q_1)$.

Definition 1. *Geodesic polygon.*
The geodesic polygon with vertices $Q_0, Q_1, ..., Q_n$ on a surface S is the piecewise curve composed by the geodesic shortest curves $c_g(Q_i, Q_{i+1})$, $i = 0, ..., n-1$.

Let

$$P^0 = \{P_0^0, P_1^0, P_2^0, ..., P_{2n-1}^0, P_{2n}^0\} \tag{10}$$

be a sequence of points on a surface S and denote by $\omega_i^0 > 0$ a local tension parameter associated to the subsequence $P_i^0, P_{i+1}^0, P_{i+2}^0$, $i = 0, 2, ..., 2n-2$. Moreover for $0 \le t \le 1$, the point $R \in c_g(Q_0, Q_1)$, such that $d_g(Q_0, R) = t\, d_g(Q_0, Q_1)$ is denoted by

$$(1-t)Q_0 \oplus tQ_1$$

Given an affinely invariant linear scheme M expressed in terms of averages, the *geodesic analogue* of M is defined in [19] as the subdivision scheme obtained replacing the linear interpolation operator $a_t(Q_0, Q_1) := (1-t)Q_0 + tQ_1$ by the geodesic interpolation operator $ga_t(Q_0, Q_1) := (1-t)Q_0 \oplus tQ_1$.

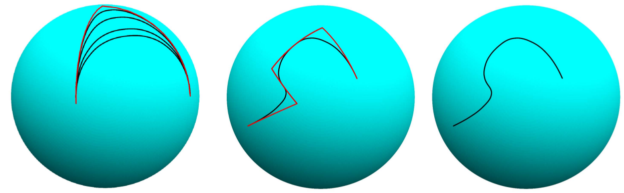

Fig. 4. Left: 3 points of on a sphere, the control polygon and the geodesic subdivision conic curve after 10 steps using $w^0 = \{0.75, 1, 2, 5\}$. Middle: The control polygon and the conic geodesic spline composed by 3 segments computed by 10 geodesic subdivision steps using $w_i^0 = 1$ for $i = 1, 2, 3$. Right: the same conic geodesic spline on the sphere.

The *geodesic conic subdivision scheme* on the surface S is defined as follows.

Geodesic conic subdivision

$$\begin{aligned} P_{2i}^{j+1} &= P_i^j && (11)\\ P_{2i+1}^{j+1} &= (1-\gamma_{2i}^{j+1})P_i^j \oplus \gamma_{2i}^{j+1}P_{i+1}^j && (12)\\ P_{2i+3}^{j+1} &= \gamma_{2i}^{j+1}P_{i+1}^j \oplus (1-\gamma_{2i}^{j+1})P_{i+2}^j && (13)\\ P_{2i+2}^{j+1} &= \frac{1}{2}P_{2i+1}^{j+1} \oplus \frac{1}{2}P_{2i+3}^{j+1} && (14) \end{aligned}$$

where the parameter γ_{2i}^{j+1} is computed using the recurrences (9) and (8), see Figure 4.

It is clear from the previous definition that the *geodesic conic subdivision scheme* (11)-(14) is the geodesic analogue of the shoulder point conic subdivision scheme (4)-(7).

Remark
The geodesic analogue of the classic conic scheme depends on the subdivision parameter t. Since geodesic curves are strongly dependent on the geometry of the surface, the limit curve generated by the geodesic analogue of the classic conic scheme is different for each value of t. Defining the geodesic conic subdivision scheme as the geodesic analogue of the shoulder point scheme has the advantage that we remove the dependence on the subdivision parameter t, thus for fixed initial polygon on S, we obtain an unique subdivision curve.

B. Convergence and smoothness analysis

Without loss of generality we restrict the analysis of the convergence to a subpolygon $P_i^0, P_{i+1}^0, P_{i+2}^0$, $i = 0, 2, ..., 2n-2$, of the initial polygon (10). To prove the convergence and the smoothness of the *geodesic conic subdivision scheme* we will use the strategy introduced in [19] for linear stationary subdivision schemes, since as remarked in [19] their sufficient conditions remain valid for non-stationary schemes, if the factors used in averaging rules are bounded. This holds true for the factors $\frac{1}{2}, \gamma_{2i}^{j+1}$ of our subdivision scheme (4)-(7). According to the results in [19], if T is a geodesic scheme analogue to an affinely invariant linear scheme M, to prove the convergence of T and the continuity of its limit curve it is enough to show that M is 0-admissible.

Given a vector of points $P = (P_i)$, we denote by $\Delta P_i = P_{i+1} - P_i$ the vector constructed as the difference of two points of vector P and we denote $\max_i \|\Delta P_i\|$ by $\|\Delta P\|_\infty$.

Definition 2. *0-admissible scheme [19]*
A linear subdivision scheme M is 0-admissible, if it is affinely invariant and fulfills the following convergence condition for all j and P^0 with a factor $\mu_0 < 1$

$$\|\Delta M^j P^0\|_\infty \leq (\mu_0)^j \|\Delta P^0\|_\infty \tag{15}$$

where $\|\Delta M^j P^0\|_\infty$ is the maximum Euclidean distance between two consecutive points of the polygon $M^j P^0$.

Since our geodesic conic subdivision scheme is the geodesic analogue of the shoulder point subdivision scheme, which is linear and invariant by affine transformations, to prove the convergence of the scheme (11)-(14) and the continuity of its limit curve, it is sufficient to show that condition (15) holds for the scheme (4)-(7). In Lemma 1 we show that the Euclidean distance between two consecutive points in the polygon of the step $j+1$ is strongly related with the Euclidean distance between two consecutive points in the polygon of the previous step. This relation is used in Proposition 1 to prove that the scheme (4)-(7) satisfies a condition like (15).

Denote by $P^j = \{P^j_{2^j i}, ..., P^j_{2^j(i+2)}\}$ the set of points on the surface S obtained applying j-times the *shoulder point conic subdivision algorithm* (4)-(7) to the subpolygon $P^0_i, P^0_{i+1}, P^0_{i+2}$, with i even.

Lemma 1. *The Euclidean distance between two consecutive points of the polygons P^j and P^{j+1} generated by the shoulder point subdivision scheme (4)-(7) are related by*

$$\begin{aligned}
\|\Delta P^{j+1}_{2i}\| &= \gamma^{j+1}_{2i}\|\Delta P^j_i\| && (16)\\
\|\Delta P^{j+1}_{2i+1}\| &\leq \left(\frac{1-\gamma^{j+1}_{2i}}{2}\right)(\|\Delta P^j_i\| + \|\Delta P^j_{i+1}\|) && (17)\\
\|\Delta P^{j+1}_{2i+2}\| &\leq \left(\frac{1-\gamma^{j+1}_{2i}}{2}\right)(\|\Delta P^j_i\| + \|\Delta P^j_{i+1}\|) && (18)\\
\|\Delta P^{j+1}_{2i+3}\| &= \gamma^{j+1}_{2i}\|\Delta P^j_{i+1}\| && (19)
\end{aligned}$$

Proof, see Figure 3
The equality (16) holds immediately from the subdivision rules (4) and (5). Analogously, we may prove the equality (19). From (5) and (7) we obtain

$$\Delta P^{j+1}_{2i+1} = \left(\frac{1-\gamma^{j+1}_{2i}}{2}\right)(\Delta P^j_i + \Delta P^j_{i+1})$$

Thus, applying the triangle inequality we get (17). Using a similar argument we obtain the inequality (18) from (6) and (7). $\diamond$

Proposition 1. *Applying j-times the subdivision rules (4)-(7) of the shoulder point conic subdivision scheme to the initial polygon $P^0_i, P^0_{i+1}, P^0_{i+2}$, with local tension parameter $\omega^0_i > 0$, it holds that there exists $\mu_0 \in (0,1)$ such that*

$$\|\Delta P^{j+1}\|_\infty \leq (\mu_0)^j \|\Delta P^0\|_\infty \tag{20}$$

Proof
Let us denote $\max\{\gamma^{j+1}_{2i}, 1-\gamma^{j+1}_{2i}\}$ by α^{j+1}_{2i}. Since $\omega^0_i > 0$, we have $0 < \gamma^{j+1}_{2i} < 1$ and this implies $0 < \alpha^{j+1}_{2i} < 1$ for $j \geq 0$.

Using the recurrence (9) - (8) it is not difficult to check that, if $\omega^0_i \geq 1$, then $0 < \gamma^0_i \leq \frac{1}{2}$ and the following inequalities hold

$$\begin{aligned}
0 < \gamma^j_i &\leq \gamma^{j+1}_{2i} < \frac{1}{2}\\
\frac{1}{2} < 1-\gamma^{j+1}_{2i} &\leq 1-\gamma^j_i < 1 && (21)\\
\alpha^{j+1}_{2i} &= 1-\gamma^{j+1}_{2i} < 1 && (22)
\end{aligned}$$

and if $0 < \omega^0_i \leq 1$, then $\frac{1}{2} < \gamma^0_i \leq 1$ and the following inequalities hold

$$\begin{aligned}
\frac{1}{2} < \gamma^{j+1}_{2i} &\leq \gamma^j_i < 1 && (23)\\
\alpha^{j+1}_{2i} &= \gamma^{j+1}_{2i} < 1 && (24)
\end{aligned}$$

Thus, for $\omega^0_i \geq 0$, from (21)-(24), we get

$$0 < \alpha^{j+1}_{2i} \leq \alpha^j_i < 1 \tag{25}$$

For any $j \geq 0$, using the relations (16)–(19) and (25) with $I^j_i = [2^j i\ ,\ 2^j(i+2)-1]$ we obtain,

$$\begin{aligned}
\max_{r \in I^{j+1}_i} \|\Delta P^{j+1}_r\| &\leq \alpha^{j+1}_{2^{j+1}i} \max_{r \in I^j_i} \|\Delta P^j_r\|\\
&\leq \alpha^{j+1}_{2^{j+1}i}\alpha^j_{2^j i} \max_{r \in I^{j-1}_i} \|\Delta P^{j-1}_r\|\\
&\dots\\
&\leq \alpha^{j+1}_{2^{j+1}i} ... \alpha^1_{2i} \max\{\|\Delta P^0_i\|, \|\Delta P^0_{i+1}\|\}\\
&\leq (\alpha^1_{2i})^{j+1} \max\{\|\Delta P^0_i\|, \|\Delta P^0_{i+1}\|\}\\
&\leq (\alpha^1_{2i})^j \|\Delta P^0\|_\infty && (26)
\end{aligned}$$

Thus condition (20) holds with $\mu_0 = \alpha^1_{2i} < 1$. $\diamond$

Theorem 1. *The geodesic conic subdivision scheme (11)-(14) with local tension parameters $\omega^0_i > 0$ applied to the initial polygon $P^0 = \{P^0_i, i = 0, ..., 2n\}$ with vertices on a C^2 continuous surface S converges to a continuous limit curve for $\|\Delta P^0\|_\infty$ sufficiently small.*

Proof
The geodesic conic subdivision scheme is the geodesic analogue of the shoulder point conic subdivision scheme. Moreover, the invariance by affine transformations and the inequality (26) means that shoulder point scheme is 0-admissible (and therefore it converges to a continuous curve [3]). Hence, the geodesic conic subdivision scheme also converges to a continuous curve for polygons P^0 such that $\|\Delta P^0\|_\infty$ is sufficiently small, see Theorem 7 in [19]. $\diamond$

In the rest of this section we focus on the proof of the C^1 continuity of the subdivision curve.

Definition 3. *Dilatation factor [4]*
Let M be a linear subdivision scheme, affinely invariant. We say that M has dilatation factor $N > 1$, if for all polygons $P = (P_i)$ and $Q = (Q_i)$, with $Q_i = P_{i+r}$ it holds

$$(MQ)_i = (MP)_{i+Nr} \tag{27}$$

The shoulder point subdivision scheme is linear and affine invariant. Moreover, it is easy to check that it satisfies the condition (27) with $r = 2$ and $N = 2$. Thus, its dilation factor is $N = 2$.

Definition 4. 1-*admissible scheme [19]*
A linear subdivision scheme M is 1-admissible, if it is 0-admissible with $\mu_0 < 1/\sqrt{N}$, where N is the dilatation factor of the scheme, and for all j and P^0, M satisfies additionally the following smoothness condition with $\mu_1 < 1$

$$N^j \|\Delta^2 M^j P^0\|_\infty \leq (\mu_1)^j \|\Delta^2 P^0\|_\infty \tag{28}$$

where $\Delta^2 P_k := \Delta P_{k+1} - \Delta P_k = P_{k+1} - 2P_k + P_{k-1}$ for a vector of points $P = (P_k)$.

Theorem 7 in [19] states that the geodesic analogue of a linear 1-admissible subdivision scheme converges to a C^1 curve for all polygons P^0 with $\|\Delta P^0\|_\infty$ small enough. Since the dilatation factor of the scheme (11)-(14) is $N = 2$, to use the previously mentioned sufficient condition we have to prove that the inequality (15) holds for $\mu_0 < \frac{1}{\sqrt{2}}$ and that there is $\mu_1 < 1$ such that (28) also holds.

Lemma 2. *Denote by $P^{j+1} = MP^j$ the polygon generated by applying the shoulder point subdivision scheme (4)-(7) to a polygon P^j. For any $\varepsilon > 0$, exists j_0 such that for all $j > j_0$ it holds*

$$\|\Delta P^{j+1}\|_\infty \leq (\frac{1}{2} + \varepsilon)\|\Delta P^j\|_\infty \tag{29}$$

Proof
After a straightforward computation using the subdivision rules (4)-(7), we get a set of relations for the first differences ΔP_m^{j+1} and ΔP_m^j. From these relations, the following inequalities hold,

$$\begin{aligned}
\|\Delta P_{2i}^{j+1}\| &= \gamma_{2i}^{j+1} \ \|\Delta P_i^j\| \\
\|\Delta P_{2i+1}^{j+1}\| &\leq (\frac{1-\gamma_{2i}^{j+1}}{2})(\|\Delta P_i^j\| + \|\Delta P_{i+1}^j\|) \\
\|\Delta P_{2i+2}^{j+1}\| &= \|\Delta P_{2i+1}^{j+1}\| \\
\|\Delta P_{2i+3}^{j+1}\| &\leq \gamma_{2i}^{j+1} \ \|\Delta P_{i+1}^j\|
\end{aligned}$$

Since $\lim_{j\to\infty} \gamma_m^j = \frac{1}{2}$, we get that for any $\varepsilon > 0$, exists j_0, such that for $j > j_0$ inequality (29) holds. ⋄

Lemma 3. *For any $\varepsilon > 0$, exists j_0, such that for all $j > j_0$, the following inequality holds for the second difference operator Δ^2 applied to the polygons $P^{j+1} = MP^j$ and P^j*

$$\|\Delta^2 P^{j+1}\|_\infty \leq (\frac{1}{4} + \varepsilon)\|\Delta^2 P^j\|_\infty \tag{30}$$

Proof
From the recursion for the first differences obtained in the previous Lemma we get,

$$\begin{aligned}
\Delta^2 P_{2i}^{j+1} &= a_0 \Delta^2 P_i^j - b_0 \Delta P_i^j, \\
\Delta^2 P_{2i+1}^{j+2} &= 0 \\
\Delta^2 P_{2i+2}^{j+2} &= a_2 \Delta^2 P_i^j - b_2 \Delta P_i^j
\end{aligned}$$

where

$$\begin{aligned}
a_0 &= \frac{1-\gamma_{2i}^{j+1}}{2}, \ \ b_0 = 1 - 2\gamma_{2i}^{j+1} \\
a_2 &= \frac{1-\gamma_{2i}^{j+1}}{2}, \ \ b_2 = 1 - 2\gamma_{2i}^{j+1}
\end{aligned}$$

Observe that $\lim_{j\to\infty} b_0 = \lim_{j\to\infty} b_2 = 0$ and furthermore $\lim_{j\to\infty} a_0 = \lim_{j\to\infty} a_2 = \frac{1}{4}$. Consequently for any $\varepsilon > 0$, exists j_0 such that for $j > j_0$ (30) holds. ⋄

Proposition 2. *For any initial polygon P^0, there exists an integer number $j_0 > 0$, such that the subdivision scheme M satisfies the following convergence condition for all $j > 0$, with factor $\mu_0 < \frac{1}{\sqrt{2}}$*

$$\|\Delta (M)^j P^{j_0}\|_\infty \leq (\mu_0)^j \|\Delta P^{j_0}\|_\infty \tag{31}$$

where $P^{j_0} = M^{j_0} P^0$.

Proof
After Lemma 2, for any given $\varepsilon > 0$, exists and integer number $j_0 > 0$, such that for all $j > 0$,

$$\|\Delta (M)^j P^{j_0}\|_\infty \leq (\frac{1}{2} + \varepsilon)\|\Delta P^{j-1+j_0}\|_\infty.$$

Since we may assume that ε has been selected, such that $\mu_0 = \frac{1}{2} + \varepsilon \leq \frac{2}{3} < \frac{1}{\sqrt{2}}$, then for all $j > 0$ holds

$$\begin{aligned}
\|\Delta (M)^j P^{j_0}\|_\infty = \|\Delta P^{j+j_0}\|_\infty &\leq \mu_0 \|\Delta P^{j-1+j_0}\|_\infty \\
&\leq (\mu_0)^2 \|\Delta P^{j-2+j_0}\|_\infty \\
&\leq ... \leq (\mu_0)^j \|\Delta P^{j_0}\|_\infty
\end{aligned}$$

⋄

Proposition 3. *For any initial polygon P^0, there exists an integer number $j_0 > 0$, such that the subdivision scheme M satisfies the following smoothness condition for all $j > 0$ with factor $\mu_1 < 1$:*

$$\|2^j \Delta^2 (M)^j P^{j_0}\|_\infty \leq (\mu_1)^j \|\Delta^2 P^{j_0}\|_\infty \tag{32}$$

Proof
After Lemma 3, for any given $\varepsilon > 0$, exists and integer number $j_0 > 0$, such that

$$\begin{aligned}
\|2 \ \Delta^2 (M)^j P^{j_0}\|_\infty &= \|2\Delta^2 P^{j+j_0}\| \\
&\leq (\frac{1}{2} + 2\varepsilon)\|\Delta^2 P^{j-1+j_0}\|_\infty
\end{aligned}$$

We may further assume that ε has been selected, such that $\mu_1 = \frac{1}{2} + 2\varepsilon < 1$. Hence, for all $j > 0$ it holds

$$\begin{aligned}
\|2^j \Delta^2 (M)^j P^{j_0}\|_\infty &= \|2^j \Delta^2 P^{j+j_0}\| \\
&\leq \mu_1 \|2^{j-1} \Delta^2 P^{j-1+j_0}\|_\infty \\
&\leq (\mu_1)^2 \|2^{j-2} \Delta^2 P^{j-2+j_0}\|_\infty \\
&\leq ... \leq (\mu_1)^j \|\Delta^2 P^{j_0}\|_\infty
\end{aligned}$$

⋄

Remark
The smallest value of j_0 such that the inequality (29) holds in the proof of Lemma 2 depends on γ_m^j and it is small, due

to the fast convergence of γ_m^j to $\frac{1}{2}$ for $j \to \infty$. For instance, if $\omega_0^0 = 20$, then after 3 iterations we already get $\gamma_0^3 = 0.52$. The inequalities (31) and (32) do not necessarily hold for the initial polygon P^0. Nevertheless, since the subdivision curve is the same starting from P^{j_0} or from P^0, we conclude from Propositions 2 and 3 that the scheme M obtained applying the subdivision rules (4)-(7) is $1-$admissible.

Theorem 2. *The geodesic conic subdivision scheme (11)-(14) with local tension parameters $\omega_i^0 > 0$ applied to the initial polygon $P^0 = \{P_i^0, i = 0, ..., 2n\}$ with vertices on a C^2 continuous surface S converges to a C^1 continuous limit curve for $\|\Delta P^0\|_\infty$ sufficiently small.*

Proof
From Propositions 2 and 3 the scheme M is $1-$admissible. Hence, from Theorem 7 in [19] we conclude that the limit curve of the geodesic analogue of subdivision scheme M is C^1 continuous, for all polygons P^0 such that $\|\Delta P^0\|_\infty$ is sufficiently small. $\diamond$

IV. The subdivision scheme on triangulated surfaces

Geodesic Bézier polynomial curves on *triangulated surfaces* were introduced in [14] by means of a subdivision algorithm which is the *geodesic analogue* of the classical de Casteljau algorithm. More precisely, for a value of $t \in [0, 1]$ previously selected and a control polygon $P^0 = \{P_0^0, P_1^0, ..., P_n^0\}$ with vertices in a triangulated surface S, the geodesic Bézier curve of degree n is defined in [14] as the limit curve of the classic Bézier subdivision applied to P^0, substituting linear interpolation by geodesic interpolation. Since geodesic curves depend on the geometry of the surface, changing the subdivision parameter t may lead to a different curve. In [14] authors select a midpoint subdivision scheme, i.e. in the step j the Bézier control polygons for the intervals $[\frac{i}{2^j}, \frac{i+1}{2^j}],\ i = 0, ..., 2^j - 1$ are computed.

In this section we use a similar approach to compute *geodesic conic curves* on triangulated surfaces, extending the method proposed in the previous section for a smooth surface to a triangulated surface. As we previously saw, unlike the geodesic Bézier curves, the geodesic conic curves don't depend on the parameter t, since the subdivision algorithm (11)-(14) is the geodesic analogue of the shoulder point subdivision scheme (4)-(7).

A. Discrete geodesic curves

The key for the implementation of the geodesic conic subdivision algorithm when S in a triangulated surface is to compute geodesic curves on S. Due to the increasing development of discrete surface models different definitions of geodesic curves on polyhedral surfaces have been introduced. Such curves are called *discrete geodesics* and we are particulary interested in *shortest geodesic curves* passing through two prescribed points.

The problem of computing shortest geodesic curves on meshes have been extensively treated, see for instance [11] and references therein. We implemented the geodesic conic subdivision scheme (11)-(14) using the method proposed in [13] to compute shortest geodesic curves passing through two prescribed points. This method is an iterative algorithm that performs the geodesic computation in two steps. The first step uses the Fast Marching Method [11] to compute an initial approximation to the shortest geodesic. The initial approximation is a polygonal curve with nodes on the edges or vertices of the triangulation. In the second step, the position of the node with the largest error is corrected and the error at neighboring nodes is updated. The process is repeated until a small error is obtained. The error at a node is computed taking into account the discrete geodesic curvature, see [14]. The position of a node on the initial approximation is corrected by unfolding a subset of faces adjacent to it.

B. Convex hull property

It is not trivial to give a proper definition of convex set in a curved geometry. However, we can find in [14] definitions of convex set and convex hull that are appropriated for the study of geodesic conic curves.

The Convex Hull property of geodesic conic curves is obtained in the same way as done in [14]. Particularly, the adaptive version of de Casteljau's algorithm relies on this property.

C. User interface and results

The geodesic conic subdivision scheme is very useful to design curves on a surface. In this section we describe how to perform the interaction with the user in an intuitive and friendly way. To obtain a smooth conic spline curve the points $P_{2i-1}^0, P_{2i}^0, P_{2i+1}^0,\ i = 0, 1, ..., n-1$ of the initial control polygon have to lie on the same geodesic curve. Since this kind of "collinearity" is not natural for the user, we introduce a simple preprocessing step. Denote by $Q_0, Q_1, ..., Q_n$ the points selected by the user on the surface S. Then, we construct the geodesic control polygon P^0 as follows,

$$\begin{aligned}
P_0^0 &= Q_0 \\
P_{2i-1}^0 &= Q_i,\ i = 1, ...n-1 \\
P_{2i}^0 &= (1-\beta_i)Q_i \oplus \beta_i Q_{i+1},\ i = 1, ...n-2 \\
P_{2n-2}^0 &= Q_n
\end{aligned}$$

where $0 < \beta_i < 1$. In other words, the vertices P_{2i}^0 are on the geodesic curve passing through P_{2i-1}^0 and P_{2i+1}^0 for $i = 1, ..., n-1$. In our experiments we set $\beta_i = 0.5$ for $i = 1, ..., n-2$ and also $w_{2i}^0 = 1$ for each segment with control polygon $P_{2i}^0, P_{2i+1}^0, P_{2i+2}^0,\ i = 0, 1, ..., n-2$. We apply the geodesic conic subdivision rules (11)-(14) and stop at some prescribed level of subdivision or when control polygons can be considered as geodesic segments. In terms of the algorithm proposed in [14] the last condition means that each control vertex has an error smaller than a prescribed tolerance.

Figure 5, 6 and 7 show the performance of the geodesic conic subdivision scheme on a triangulated surface and the advantages of using this kind of curves:

- **local control**: changing the position of any vertex of the control polygon only affects at most two segments of the geodesic conic spline.
- **geometric handles**: the weight $w_i^0 > 0$ is a geometric handle that allows to control the geometry of the section of the spline with control polygon $P_i^0, P_{i+1}^0, P_{i+2}^0$. A value of w_i^0 close to 0 generates a conic subdivision segment close to the curve $c_g(P_i^0, P_{i+2}^0)$. On the other hand, a large value of $w_i^0 > 0$ produces a subdivision segment close to the geodesic polygon with vertices $P_i^0, P_{i+1}^0, P_{i+2}^0$.

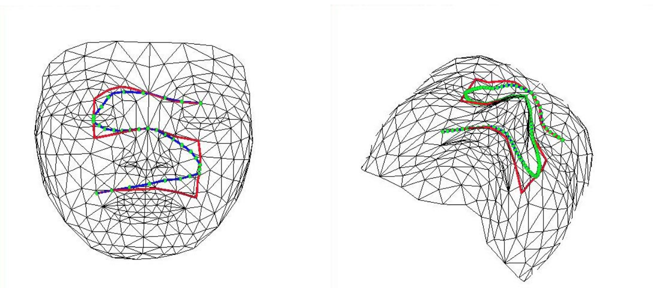

Fig. 5. The initial geodesic control polygon (red) on a triangulated surface and the vertices (green) of the geodesic conic subdivision curve with all weights equal to 0.5. Left: after 3 steps , Right: after 6 steps

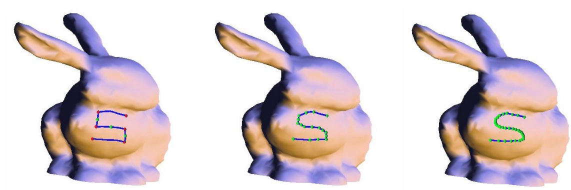

Fig. 6. Left: Initial control polygon on a triangulated surface (in red the points $Q_0, ..., Q_5$, in blue the points P_2, P_4, P_6) middle and right: the geodesic subdivision curve of first and second steps with all weights equal to 1.

Fig. 7. Initial polygon on a triangulated surface and the geodesic conic subdivision curves obtained with three values of the weight, $w_0^0 = 0.5, 1, 4$.

V. Conclusions

A new subdivision scheme for designing curves on surfaces has been proposed. It can be considered as a natural generalization of conic Bézier curves. The limit curve of this scheme is a C^1 continuous curve if the surface is C^2 continuous. The scheme depends on free parameters that are very useful to control the shape of the subdivision curve, which also enjoys the convex hull property. These geometric handles make the curves generated for the proposed scheme a suitable tool for designing, editing and trimming on surfaces.

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