Beta-connection: An Approach to Generate Families of Models from Planar Sections

ALEX JESUS QUADROS VARGAS, LUIS GUSTAVO NONATO, ROSANE MINGHIM, MARIA CRISTINA F. DE OLIVEIRA

ICMC - Instituto de Ciências Matemáticas e de Computação, USP, Campus de São Carlos
C.P. 668, 13560-970, São Carlos, SP Brasil
{acvargas, gnonato, rminghim, cristina}@icmc.sc.usp.br

Abstract. A new three-dimensional reconstruction technique is presented that uses an integer control parameter, denoted β, to produce a family of models from a given set of planar cross-sections. Parameter β supports multiple choices for solving the correspondence problem, i.e., the problem of deciding which regions from two consecutive cross-sections must be connected into a single component. Thus, unlike current reconstruction methods, the beta-connection algorithm enables the consideration of multiple alternatives when establishing region correspondence. In addition to this flexibility, which is useful in creating models with complex topologies, the algorithm produces PL-manifolds and respects the re-sampling condition, thus providing an interesting reconstruction solution for many practical visualization and numerical simulation applications.

1. Introduction

Many algorithms have been developed to reconstruct three-dimensional models from a set of parallel planar cross sections of an object. Such algorithms must handle three intrinsic reconstruction problems, namely, correspondence, branching, and tiling. The correspondence problem arises when it is necessary to decide which regions in two adjacent slices must be connected into a single component. Branching is related with handling the saddle points that may appear in the models. Tiling refers to the construction of a polygonal mesh connecting regions in adjacent sections.

Reconstruction algorithms described in the literature, albeit adopting different strategies to solve the correspondence problem, share a common characteristic, which is the lack of flexibility in defining the correspondence amongst regions in consecutive slices. In deciding which regions must be connected, most techniques either assume a previously established correspondence, or adopt some proximity criterion. Thus, from the set of possible choices, only one is considered in creating the 3D model. However, flexibility in the choice of correspondence is important because the information given by the cross section is not sufficient to guarantee that a 'correct' connection has been made. This is illustrated in Figure 1, where the regions in the slices of Figure 1(a) could be obtained from any of the objects shown in Figures 1(b), 1(c) or 1(d).

This work extends a previous technique for volume reconstruction from planar sections [12] to allow consideration of multiple choices when defining a correspondence amongst multiple regions. The motivation for a more flexible strategy to treat the correspondence problem came from applications in reconstruction of arterial structures and creating solid models of objects with complex topologies. In those applications, the heuristic adopted by our previous algorithm [12] failed to produce the desired models.

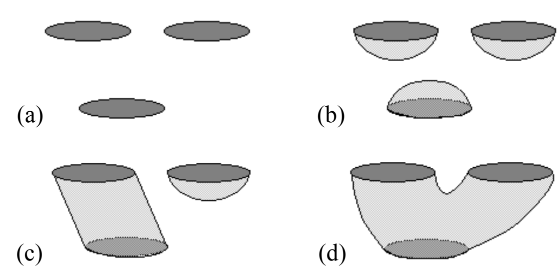


Figure 1 Models with identical planar cross sections.

The algorithm presented in this paper is particularly adequate to the reconstruction of objects with high degree of branching and/or merging. Based on the 3D Delaunay Triangulation, it produces a family of models from a given set of cross sections. Model generation is governed by an integer parameter, denoted β, whose value controls the degree of proximity to be assumed when establishing correspondence amongst regions in neighboring slices. Increasing values of β result in stronger region connection. The resulting models are guaranteed to be 3D piecewise linear manifolds (PL-manifolds), a highly desirable property for numerical simulations, which is one of our target applications. The algorithm also satisfies the re-sampling criterion, i.e., intersection of the resulting model with the original cutting planes produces the original set of planar cross-sections.

This paper is organized as follows: Section 2 provides a brief overview of related work in surface and volume 3D reconstruction from planar cross-sections. Section 3 reviews relevant definitions and properties of Delaunay triangulations and Voronoi diagrams. Section 4 describes the Beta-connection approach. The subdivision and disconnection processes employed in model generation are presented in Section 5. Section 6 presents the algorithm and a complexity analysis. Section 7 shows some illustrative examples, and finally conclusions and further work are presented in Section 8.

2. Related Work

Many non-invasive techniques, such as MRI (Magnetic Ressonance Imaging), CT (Computer Tomography), and laser microscopy, produce cross sections of objects from which 3D models can be reconstructed that approximate the original objects. Reconstruction has applications ranging from education to diagnosis and simulation of medical procedures. Some of the strategies described in the literature build the 2D surface that bounds the object, while others produce volumetric representations, typically using voxels or tetrahedrons as primitives. Some of these strategies are briefly reviewed in this section, with emphasis in the approach they adopt to solve the correspondence problem.

Implicit techniques create a smooth implicit function whose zero set coincides with the boundaries of the regions contained in the slices. Jones & Chen [2] use a signed distance field to create the implicit function, and then apply the Marching Cubes algorithm [3] to extract the surface bounding the regions. Correspondence amongst regions in different slices is automatically defined by the implicit function, making it difficult to choose specific connections.

Voxel based approaches generate a representation of the original object directly from the image data without building a surface or volumetric mesh [4]. Although it avoids critical issues, such as segmentation of contours in the slices and management of the mesh, this approach does not allow control of the correspondence amongst contours, which is actually determined by the interpolation approach adopted to create the visualization.

The so-called optimal approaches employ graph theory to build the 3D model. Shinagawa & Kunii [5] expand a discrete toroidal graph to obtain a continuous graph, and homotopy is used to reconstruct parametric surfaces and define the correspondences. Meyers et al. [6] assemble contours into cylinders and use a minimum spanning tree to establish the correspondence amongst them. In deformable models deformations are successively applied to an initial model to gradually approximate the original object [7]. Different connections may be obtained from distinct initial models, and finding suitable initializations is not a trivial task.

Many algorithms employ heuristics to construct a 3D model from planar cross sections. Ekoule et al. [8] map boundaries of regions into their convex hull and use the shortest edge to connect convex hulls that are in adjacent slices. Bajaj et al. [9] define three criteria to be met by the reconstructed surface and build an algorithm based upon them. In regions where the conditions cannot be satisfied the surface is completed with a triangulation that takes into account the Voronoi skeletons. Correspondence amongst regions is implicitly defined by the reconstruction criteria and a volumetric model may also be created with a tetrahedralization process [10].

Boissonnat [11] uses the projections of the 2D Voronoi diagrams of every pair of consecutive slices to build a graph that embeds information on the 3D Delaunay triangulation. From that graph a volumetric model is generated through tetrahedron elimination. Correspondence is given by the Delaunay triangulation and the tetrahedron elimination.

Nonato et al. [12] use the 3D Delaunay triangulation to build a volumetric model that is a manifold. Although the algorithm accepts previously defined connections, correspondence amongst regions is automatically determined by a proximity criterion. This criterion verifies the occurrence of a special type of tetrahedron, called reverse tetrahedron, in the Delaunay triangulation. In this paper we extend our previous solution [12] with an algorithm that can create a family of possible 3D models from a given set of planar cross sections. Unlike existing algorithms, the technique considers multiples choices for the correspondence problem. It also produces PL-manifolds and satisfies the re-sampling condition.

3. Basic Concepts

In this section we briefly introduce some definitions and properties of Delaunay triangulations and Voronoi diagrams. Detailed descriptions on both, as well as on algorithms to build them, can be found elsewhere [14, 15]. Voronoi skeletons, tetrahedron classification and strong contour overlapping are also introduced.

3.1 Voronoi Diagram and Delaunay Triangulation

Let $A=\{x_1,...,x_n\}$ be a set of points in general position in R^m, i.e., there is no affine subspace of R^m containing A and there is no sphere S^{m-1} through a subset of A with $m + k$, $k>1$ points. In this text sets of points are always assumed to be in general position. The Voronoi diagram for A is a decomposition of R^m into m-dimensional convex cells $V_1,...,V_n$ with the following properties:

1. Each V_i contains a single point x_i of A.

2. Given $x\in R^m$, $x\in V_i$ if and only if $d(x,x_i) \leq d(x,x_j)$, for every $i \neq j$, where d is the Euclidean distance.

It can be shown that the intersection of k Voronoi cells, $2 \leq k \leq m+1$, is either empty or is an $(m-k+1)$-dimensional cell contained in the diagram. A triangulation can be obtained from the Voronoi diagram by associating each of its p-dimensional cells with an $(m-p)$-simplex. Considering, for example, the 2D case, each vertex in the diagram (given by the intersection of three cells) is associated with a triangle (2-simplex); each edge is associated with an edge of the triangulation (1-simplex); and each cell V_i is associated with a vertex x_i of the triangulation (0-simplex). The triangulation so obtained is called Delaunay triangulation, and it maintains with the Voronoi diagram a duality relationship.

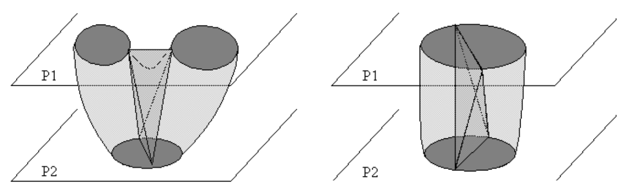
From the duality relationship and the general position of the points it follows that each 0-dimensional cell of the diagram is the center of a sphere circumscribing an m-simplex and that this sphere does not contain in its interior any other points of A. If the points in A are positioned on two consecutive parallel planar sections P_1 and P_2, the intersection of their 3D Delaunay triangulation with P_i ($i = 1,2$) is the 2D Delaunay triangulation of the points in P_i. This property was proved by Boissonnat [11] and will be useful in the following sections.

Another important fact demonstrated by Boissonnat [11] is that, if the points in P_1 and P_2 are polygons vertices, some edges of such polygons may not belong to the Delaunay triangulation. Boissonnat has shown that a polygon edge that is not in the Delaunay triangulation may be subdivided into new edges that will be contained in the triangulation. Based on the above results, from now on, the boundaries of the regions contained in the set of parallel planar cross sections will be defined by the non-intersecting polygons whose edges belong to the Delaunay triangulation of its vertices. The polygons satisfying this condition are called contours.

3.2 Voronoi Skeletons

Let C_1 and C_2 be two sets of contours bounding regions contained in adjacent planar sections P_1 and P_2 and DT the 3D Delaunay Triangulation of the vertices in $C_1 \cup C_2$. Suppose that contours in C_1 and C_2 are oriented in such a way that the interiors of the regions are always on their left-hand side. Based on contour orientation, an edge of DT contained in P_1 or P_2 can be classified as either internal or external to a region. Edges on the contours are labeled as contour edges. Note that only the edges contained in P_i ($i = 1,2$) are classified, those DT edges lying between P_1 and P_2 are not classified.

DT has tetrahedrons of two types: those with a face in P_1 (P_2) and a vertex in P_2 (P_1), named 'type 1' tetrahedrons, and those with a edge in P_1 an another edge in P_2, named 'type 2'. Based on the classification of its edges, a tetrahedron in DT may be classified as internal, external or redundant. A tetrahedron with at least one edge internal to a region r is said to be an internal tetrahedron of r. A tetrahedron with at least one external edge and no internal ones is said to be an external tetrahedron, and a 'type 2' tetrahedron with two contour edges is called a redundant tetrahedron. A special case is the reverse tetrahedron [12], a 'type 2' tetrahedron whose edges in P_1 and P_2 are either both internal ones, or are one internal and the other external, as shown in Figure 2. Note that a reverse tetrahedron can be internal to two regions simultaneously. Reverse tetrahedrons bear a straight relation with the relative positioning of regions in adjacent slices, as discussed by Nonato et al. [12].



(a) (b)

Figure 2 Reverse tetrahedrons.

Let VD_i be the 2D Voronoi diagram of the vertices of C_i in P_i ($i = 1,2$). The Voronoi skeleton of a region $r \subset P_i$ is defined as the subset of the VD_i edges that are dual to the DT internal edges with ends in the contours of C_i that bound r. Figure 3 shows a set of regions and their correspondent Voronoi skeletons.

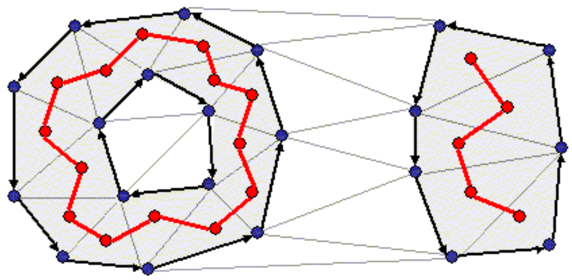


Figure 3 Voronoi skeletons.

Two regions $r_1 \subset P_1$ and $r_2 \subset P_2$ are defined as strong overlapping regions if the orthogonal projection of the Voronoi skeleton of r_1 onto P_2 intersects the Voronoi skeleton of r_2. The relevance of strong overlapping regions will become apparent in Section 4, where the beta-connection family is introduced.

4. Beta-Connection

The beta-connection approach tackles the reconstruction problem by handling pairs of consecutive slices. Thus, from now on we assume an input set of points composed by vertices of contours lying on two parallel planar sections. This Section starts with a description of a graph structure from which beta-connections are derived.

4.1 The DT Associated Graph

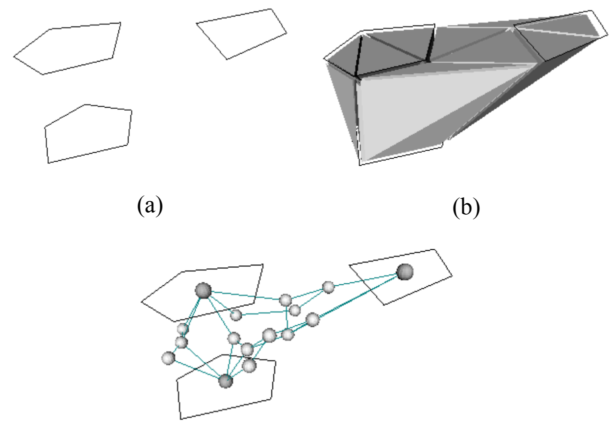
Let C_1 and C_2 be two sets of contours representing the boundaries of regions R_1 and R_2 contained in adjacent planar sections P_1 and P_2, respectively, and DT the 3D Delaunay triangulation of vertices in $C_1 \cup C_2$. From DT, a graph G can be constructed that has two types of nodes. Each region $r \in R_1 \cup R_2$ and all its internal tetrahedrons define a so-called called *region node* in G, and each external or redundant tetrahedron of DT define an *external node* in G. Edges linking pairs of nodes in G are obtained as follows:

1. If two regions r_1 and r_2 have internal tetrahedrons with a common face, or if there is a reverse tetrahedron with internal edges in r_1 and r_2, then the region nodes representing r_1 and r_2 are connected by an edge.

2. If an external or redundant tetrahedron t share a face with an internal tetrahedron of a region r, then the region node r and the external node t are connected by an edge.

3. Two external nodes are connected if and only if the tetrahedrons they represent share a face.

Figure 4 illustrates a set of regions and the graph G generated from their DT. It follows from the above construction process that graph G is connected. Moreover, each of its external nodes has a minimum degree equal to 1 and a maximum degree equal to 4.



(a) (b) (c)

Figure 4 a) Original contours; b) 3D Delaunay Triangulation; c) Graph.

A path of length n connecting two region nodes a and b in G is a sequence of nodes $\{\tau_1,\dots,\tau_{n+1}\}$, such that $\tau_1 = a$, $\tau_{n+1} = b$, and (τ_i, τ_{i+1}), $i = 1,\dots,n$, is an edge of G. The distance between two region nodes a and b, denoted $d_{\mathsf{G}}(a,b)$, is the length of the shortest path between a and b.

4.2 Beta-Components

Let β be a natural number. Two region nodes a and b are said β-connected, denoted $a \approx_\beta b$, if there is a sequence of region nodes $\{\sigma_1,\dots,\sigma_k\}$, where $\sigma_i \in \mathsf{G}$ $(i = 1,\dots,k)$, $\sigma_1 = a$ and $\sigma_k = b$, such that $d_{\mathsf{G}}(\sigma_i,\sigma_{i+1}) \le \beta$. It can be shown that relation $\approx_\beta$ is an equivalence relation.

The equivalence relation property guarantees that each value of β defines equivalence classes constituted by those region nodes of G positioned at a distance smaller than or equal to β from one another. As each region node represents a region contained in a plane P_i $(i= 1,2)$, β also defines equivalence classes for the original regions, thus allowing correspondence to be specified through equivalence classes. In other words, each equivalence class defines which regions are to be connected. Different values of β produce different equivalence classes, enabling multiple choices of correspondence amongst the regions.

An important property of β-connection, which is proved in [13], describes the behavior of the connection process as β increases. This property states that strong overlapping regions are the first to be connected, i.e., for any $\beta \geq 1$ they will be placed into a single component and thus connected. This is reasonable because these are regions with a high degree of overlapping. Connection of regions with strong overlapping is the approach adopted by most 3D reconstruction algorithms, as it is natural to place those regions into a single component. However, for those algorithms this is the only possible criterion, whereas in beta-connection this is the starting point when considering possible connections.

Note that for $\beta = 0$, no correspondence amongst regions is established and that the number of regions grouped into a single equivalence class increases as β increases. All regions will be contained in a single class for a sufficiently large β.

The above reasoning shows that strong overlapping regions share reverse tetrahedrons and are always the first ones to be grouped when β increases from zero. Another important property that follows from the propositions presented by Nonato et al. [12] is that there is no reverse tetrahedron internal to two regions, unless these are strong overlapping regions. This is useful in the disconnection of components presented in Section 5.

5. Component Disconnection and the Subdivision Process

As described above, each equivalence class originates a connected component in the reconstructed model. As the Delaunay triangulation is a connected simplicial complex, the tetrahedrons linking the components must be removed in order to disconnect them. This tetrahedron elimination process is described in the following.

5.1 Connected Components Generation

The connected components are computed from the 3D Delaunay triangulation and a given value for the β parameter. At this stage, all connected components have already been defined, but are still part of the whole triangulation. To disconnect components according to the equivalence classes specified by β it is necessary to remove from the Delaunay triangulation all external and redundant tetrahedrons whose vertices are not in the same component.

Although tetrahedron elimination is necessary, it may not be sufficient to disconnect the components. In some situations, as illustrated in Figure 5, components may remain connected through vertices or contour edges after eliminating the tetrahedrons. In this case, vertices (and edges) must be displaced (translated) to completely disconnect components, as shown in Figure 5(c). For additional information on how this operation is performed see Nonato et al. [12].

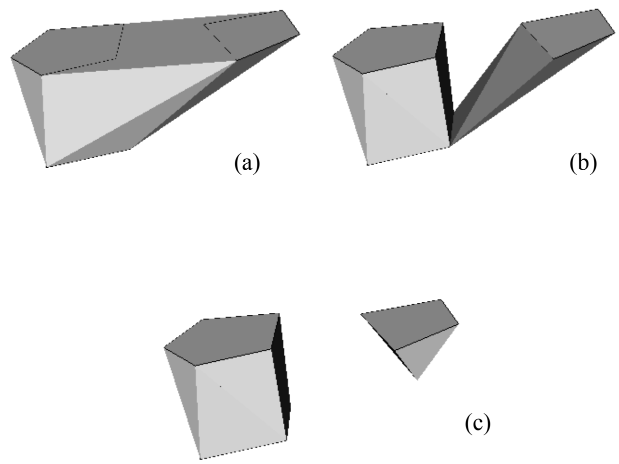


Figure 5 a) 3D Delaunay Triangulation; b) Elimination of external tetrahedron; c) Translation of vertices.

A special case deserving attention is $\beta = 0$, when each region generates an independent connected component. In this case the presence of reverse tetrahedrons in strong overlapping regions does not allow disconnection to be properly executed. The problem is that if vertices are translated to disconnect strong overlapping regions, reverse tetrahedrons internal to both regions will remain in one of them and generate “wedges” in the internal edges of the other region (Figures 6(a) and 6(b)).

A wedge characterizes a singularity, i.e., the boundary of the union of the tetrahedrons around the wedge is not homeomorphic either to a sphere or to a half-sphere. Singularities must be avoided in order to ensure that the reconstructed object is a PL-manifold. Thus, to disconnect strong overlapping regions it is necessary to duplicate the reverse tetrahedrons and then translate vertices and edges, as shown in Figure 6(c).

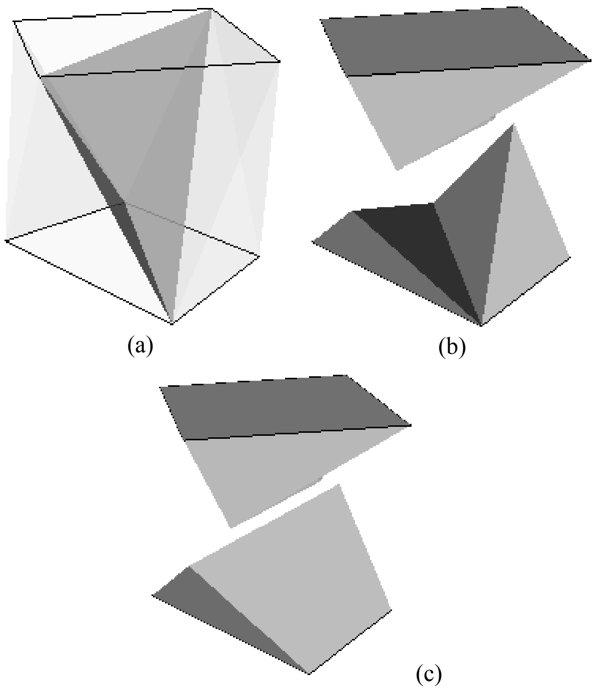


Figure 6 Duplicating the reverse tetrahedron.

5.2 Tetrahedron Subdivision

To ensure that the connected components obtained from the disconnection process satisfy the re-sampling condition, external tetrahedrons in these components must also be eliminated. However, as in the disconnection case, this may introduce singularities. This is solved with an appropriate tetrahedron subdivision process (TSP, for short), originally introduced by Nonato et al. [12]. It consists of three steps. First, external tetrahedrons whose removal does not introduce singularities are eliminated. Then, all the external tetrahedrons whose elimination introduces singularities are subdivided. Finally, the new vertices introduced in the subdivision are translated to an intermediate position between the consecutive slices to ensure that each component meets the re-sampling criterion.

Tetrahedron subdivision follows two criteria. For *type 1* tetrahedrons, a new vertex is inserted in each external edge. The number of new tetrahedrons introduced depends on the number of new vertices created: two tetrahedrons if one new vertex was created, and three or four new tetrahedrons if two or three vertices were created, respectively. For *type 2* tetrahedrons, two or four new tetrahedrons may be introduced, depending on whether one or two new vertices are inserted into the external edges. This subdivision process guarantees the manifold condition. The approach is fully described and illustrated elsewhere [12]. Figure 9 shows the TSP results for a set of four regions. Note that the branching problem was solved in a very satisfactory manner.

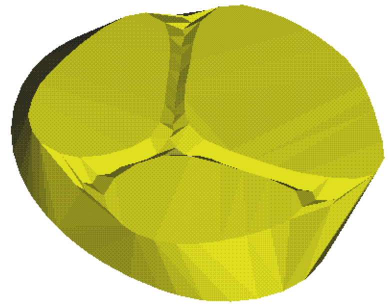


Figure 9 Result of the tetrahedron subdivision process.

6. The Algorithm

Let *C* be a set contours bounding the regions contained in two adjacent planar sections and β a natural number. The Beta-connection reconstruction algorithm can be stated as follows:

(1) *Delaunay triangulation*: Compute the 3D DT of vertices in *C*. Classify tetrahedrons as internal, external, reverse or redundant.

(2) *Beta-components*: Create 3D DT associated graph and compute Beta-components from the equivalence classes defined by β.

(3) *Connected components*: Remove external and redundant tetrahedrons connecting the Beta-components. If necessary, translate vertices and duplicate reverse tetrahedrons.

(4) *Reconstruction:* For each Beta-connection:

   a) Remove external and redundant tetrahedrons whose elimination does not introduce singularities.

   b) Subdivide the remaining external tetrahedrons.

   c) Translate the new vertices to an appropriated position between slices.

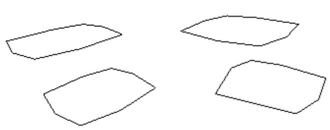
This algorithm performs geometrical computations only when deriving the 3D Delaunay triangulation. All the remaining operations and tests are topological, ensuring a more robust and efficient implementation.

Since the classification, elimination, and subdivision steps can be done in linear time on the number of tetrahedrons, the computational cost of the algorithm is governed by the construction of the Delaunay triangulation and by the computation of the Beta-components. The 3D Delaunay triangulation may be built in $O(n^2)$ with an incremental algorithm [14], where n is the number of vertices. Computation of Beta-components reduces to computing, for each region node $\sigma_i \in$ G, the distances from σ_i to all the region nodes $\sigma_j \in$ G, $i \neq j$. This computation can be done with a breadth-first search algorithm [16] in $O(N_r E)$, where N_r and E are the number of region nodes and the number of edges in G, respectively.

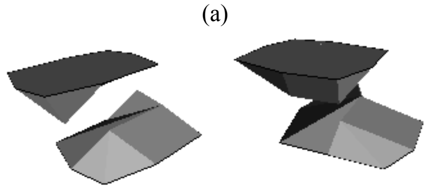
7. Results and Examples

This section presents some models obtained with the Beta-connection algorithm. For visualization purposes the *Visualization Toolkit* (VTK) [17] was used in connection with a C++ implementation of the algorithm described in the previous section. These illustrative visualizations were produced to emphasize the quality of the models and the importance of flexibility in choosing correspondence amongst regions.

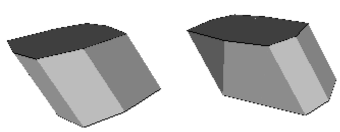
Figure 10(a) presents a set of regions in two adjacent planar cross sections, and Figures 10(b) to 10(d) show the models generated with β values equal to 0, 1 and 3, respectively. The model obtained with β = 2 would be the same depicted in Figure 10(c), as in the underlying graph G the contours positioned at a distance given by β = 2 are the same ones positioned at a distance given by β = 1.



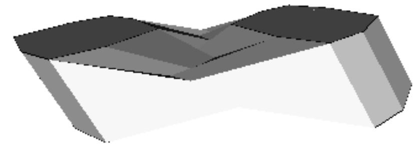
(a)



(b)



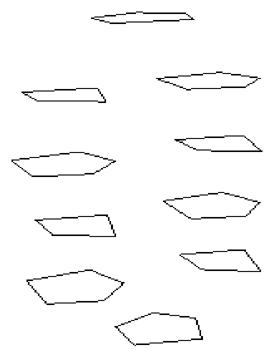
(c)



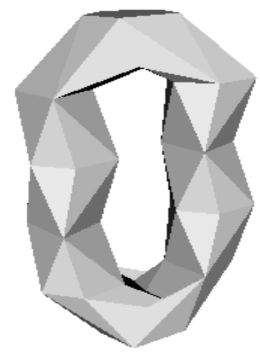
(d)

Figure 10 a) Original contours; b), c) and d) models generated with β equal to 1, 0 and 3 respectively.

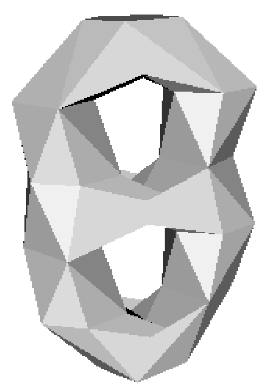
The example in Figure 11 illustrates the importance of flexibility in establishing connections amongst components. Figure 11(a) presents a set of regions contained in a sequence of slices. Because the reconstruction process analyses two adjacent planar sections at a time, it is possible vary the values of β across different pairs of slices, a capability that makes Beta-connection a powerful tool in solid modeling. Figures 11(b) to 11(d) show three different models obtained by assigning different values of β to each pair of slices. Observe that the topology of the resulting models is completely different in each case.



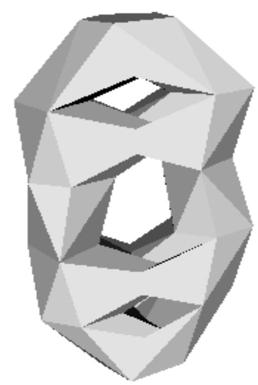
(a)



(b)



(c)



(d)

Figure 11 Models generated with different values of β when connecting components in different pairs of consecutive slices.

8. Conclusions

We present a novel technique for handling the connection problem in reconstructing 3D structures from planar cross sections. This technique offers flexibility in choosing how components are to be connected, allowing multiple choices to be considered, and solves the branching and tiling problems in a very satisfactory manner. The algorithm manages to generate a one-parameter family of models from a single set of samples based on an approach that computes the distance amongst regions through a Delaunay triangulation. Resulting volumetric models are robust PL-manifolds useful in solid modeling, numerical simulation and visualization applications.

We are particularly interested in applying this approach in the visualization of the arterial system of the liver and in representing volumetric meshes in numerical simulations. The fact that the resulting mesh is free of singularities simplifies computations in the simulation of physical phenomena. For some numerical simulations performed, such as heat transfer, little re-meshing was necessary to comply with requirements of the numerical methods. An overall re-meshing technique specific for meshes produced with this algorithm is under development.

Acknowledgements

The authors wish to acknowledge the support of FAPESP, the state of São Paulo Research Funding Agency, and CNPq, the Brazilian Federal Research Funding Agency.

References

[1] H. Edelsbrunner, E.P. Mucke, "3-dimensional alpha-shapes", *ACM Trans. Graphics* 13(1) (1994), 43-72.

[2] M.W. Jones, M. Chen, "A new approach to the construction of surfaces from contour data", *Proc. EUROGRAPHICS'94* 13(3) (1994), 75-84.

[3] W.E. Lorensen, H. E. Cline, "Marching cubes: a high resolution 3D surface construction Algorithms", *ACM SIG. Comp. Graphics* 21 (1987), 163-169.

[4] A. Kaufman, "Advances in Volume Visualization", *SIGGRAPH'98* Course Notes 24, Orlando, 1998.

[5] Y. Shinagawa, T.L. Kunii, "The homotopy model: a generalized model for smooth surface generation from cross sectional data", *The Visual Computer* 7 (1991), 72-86.

[6] D. Meyers, S. Skinner, K. Sloan, "Surface from Contours", *ACM Trans. Graphics* 11(3) (1992), 228-258.

[7] A. Singh, D. Goldgof, D. Terzopoulos (eds.), D*eformable models in medical image analysis*, IEEE CS Press, 1998.

[8] A.B. Ekoule, F.C. Peyrin, C.L. Odet, "A triangulation algorithm from arbitrary shaped multiple planar contours", *ACM Trans. Graphics* 10(2) (1991), 182-199.

[9] C.L. Bajaj, E.J. Coyle and K. Lin, "Arbitrary topology shape reconstruction from planar cross sections", *Graphical Models & Image Processing* 58(6) (1996), 524-543.

[10] C.L. Bajaj, E.J. Coyle and K. Lin, "Tetrahedral meshes from planar cross-sections", *Comput. Methods Appl. Mech. Engineering* 179 (1999), 31-52.

[11] J-D. Boissonnat, "Shape reconstruction from planar cross sections", *Computer Vision, Graphics and Image Processing* 44 (1988), 1-29.

[12] L.G. Nonato, R. Minghim, M.C.F. Oliveira and G. Tavares, "A novel approach for Delaunay 3D reconstruction with a comparative analysis in the light of applications", *Computer Graph. Forum* 20(2) (2001), 161-174.

[13] L.G. Nonato, A.J.C. Vargas, R. Minghim, M.C.F. de Oliveira, "Beta-Connection: A family of models generated from planar sections", Cadernos de Computação ICMC-USP, 2(2) (2001), 117-138.

[14] S. Fortune, "Voronoi diagrams and Delaunay triangulation", in D-Z. Du, F.K. Hwang (eds.), *Computing in Euclidean Geometry*, Wor1d Scientific Pub. Co. (1994), 193-233.

[15] F. Aurenhammer, "Voronoi diagrams – a survey of a fundamental geometric data structure", *ACM Comp. Surveys* 23(3) (1991), 345-405.

[16] T.H. Cormen, C.E. Leiserson, R.L. Rivest, *Introduction to algorithms,* MIT Elect. Engineering and Computer Science Series, 1999.

[17] W.J. Schröeder, K. Martin and W. Lorensen. T*he visualization toolkit, an object-oriented approach to 3d graphics*, 2nd ed., Prentice-Hall, 1998.