Curvature Operators in Geometric Image Processing

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Abstract. In this work we study the problem of reconstructing an image from a perceptual segmentation based on a geometric classification of its points using non-linear curvature filters. We give a mathematical proof that an image can be reconstructed from the regions of non-zero gaussian curvatures. This result provides the theoretical background for a new theory of non-linear two dimensional signal processing as proposed by C. Zetzche, E. Barth and B. Wegmann ([15, 16]). We use curvature measures to detect edges and vertices (roughly two dimensional regions) and show that reconstruction is possible from these elements.

Keywords: Image coding; image reconstruction; image processing; image compression.

1 Introduction

Linear operators are frequently used in vision and image processing. These operators are usually called image transforms. Different transforms have appeared in the literature from the ubiquitous Fourier transform to the wavelet transform.

The success of the linear methods in image processing comes from the simplicity in computing with these transforms. The computation of the inverse of a transform, even though being useless for some applications, plays a fundamental role in applications such as image coding and compression.

We should mention that non-linear methods have appeared in the literature of image processing. Among them we could mention the well known fractal image compression technique. In this method the image is associated to a unique fixed point of an iterated function system (IFS). The IFS parameters are used to code the image, and the original image can be recovered from the action of the IFS on some initial image, which is a non-linear process.

1.1 Importance of Non-linear filters

In spite of the successful use of linear methods in digital image processing and in modeling some functions of the visual system [4, 3], it is well known that some of these systems contain detectors which are insensitive to image features whose variation is zero or unidirectional [9].

It is not difficult to realize that linear filters can not offer an adequate model to receptive cells that are insensitive to stimulus that have no variation or to those that posses only one-dimensional variation. This happens because the eigenfunctions of the filter are one-dimensional signals [15, 16]. Experiments reveal that most of the information carried by an image is located in the points which are extremes of the curvature [1].

Such cells can not be modeled by linear filters, therefore the study of non-linear operators for image processing becomes a necessity. In [16], C. Zetzsche, E. Barth and B. Wegmann present a detailed study of the limitations of linear filters and propose the use of the Gaussian curvature of the Monge surface associated to the image as a good non-linear operation for perceptual studies. Therefore the study of this operator and its inverse assumes a fundamental role in their proposal.

The Gaussian curvature as a non-linear filter is studied with more details in [2], where it is shown that it is possible to obtain a partial reconstruction from a set of points of non-zero Gaussian curvature. Also, the authors conjecture that this set of non-zero Gaussian curvature contains the necessary information to reconstruct the image completely. A proof of this fact is the main result of this article.

We should remark that the discussion above is closely related to the problem of reconstructing an image from its edges, as stated in the famous conjecture by D. Marr [12]. Reconstruction of images from a subset of points, such as the edges, can provide efficient compact representation of images. This problem has been addressed recently in [6].

2 Differential Geometry and Image

We will work with grayscale image. A grayscale image can be modeled as a function $h\colon U \to \mathbb{R}$, where the *image support* U is a subset of the plane, and $h(u, v)$ represents the gray value at the point (u, v). In order to use techniques from differential geometry we will need to differentiate the image. This presents no problem with the use of scale-space regularization techniques [10]. In this case we consider the representation of the image h at some proper scale σ, using a convolution with some gaussian filter of median 0 and variance σ:

$$h_\sigma = h * g_\sigma,$$

where

$$g_\sigma(u, v) = Ke^{-\frac{u^2+v^2}{\sigma^2}}.$$

The parameterization of the Monge surface of the image is given by

$$x(u, v) = (u, v, h(u, v)).$$

The first and second fundamental forms, [5], are given by

$$C = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 + h_u^2 & h_u h_v \\ h_u h_v & 1 + h_v^2 \end{pmatrix} \tag{1}$$

and

$$B = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \frac{1}{\sqrt{1 + h_u^2 + h_v^2}} \begin{pmatrix} h_{uu} & h_{uv} \\ h_{uv} & h_{vv} \end{pmatrix} \tag{2}$$

From G and B we can compute easily the *mean* and *gaussian* curvatures of the image using the equations

$$K = \frac{\det B}{\det C} \tag{3}$$

and

$$H = 1/2 \, \mathrm{trace}(BC^{-1}). \tag{4}$$

The points of the image surface are classified according to the sign of K and H:

1. Elliptic if $K > 0$;
2. Hyperbolic if $K < 0$;
3. Parabolic if $K = 0$ and $H \neq 0$;
4. Planar if $K = H = 0$.

3 Segmentation

If S is a surface, and X is a subset of S, we will denote by $\mathrm{int}(X)$, $\overline{X}$ and ∂X the set of interior points, the topological closure and the boundary of X respectively.

The operation that associates to X the topological closure of its interior points is called *regularization* and will be denoted by $\mathcal{R}(X)$, that is $\mathcal{R} = \overline{\mathrm{int}(X)}$. A set X is *regular* if $\mathcal{R}(X) = X$. A regular set X is called a *region* if its boundary consists of a piecewise differentiable curve.

Definition. A segmentation of a region $R \subset S$ is a collection $\mathcal{S} = \{U_\lambda\}$ of subsets of R such that

1. $R = \bigcup U_\lambda$;
2. If $\alpha \neq \beta$, then $\overline{U}_\alpha \bigcap \overline{U}_\beta$ has empty interior.

Each subset U_λ is called a *segment* of U. A segmentation is finite if it possesses a finite number of segments. A segmentation is *locally finite* if each point has a neighborhood that intersects only a finite number of segments. A segmentation of an image consists of the segmentation of the associated image surface.

We should remark that if $\mathcal{S} = U_\lambda$ is a segmentation of a region U, then if $\overline{\mathcal{S}} = \overline{U}_\lambda$ is a segmentation of U using closed sets. Also, it is true that any segmentation of a region U can be regularized. A proof of this fact can be found in [14].

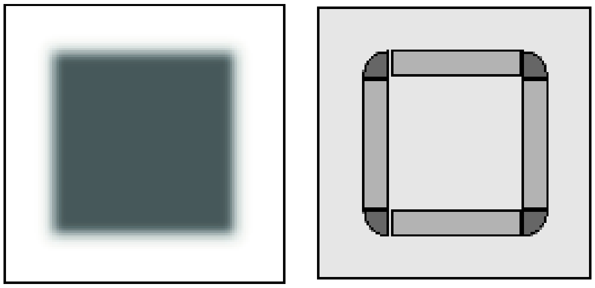
3.1 Perceptual segmentation

An image has a huge amount of redundant data, and this redundancy is exploited in the different image coding techniques. This redundancy has been classified in [16]. A description will be given below.

Definition (Perceptual segmentation). We classify the points of a grayscale image $h\colon U \to \mathbb{R}$ according to the following criteria:

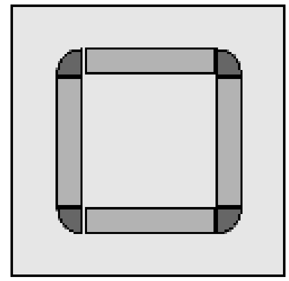
1. $p \in 0D$ if h is constant in a neighborhood of p;
2. $p \in 1D$, if $p \notin 0D$ and there exists a decomposition of a neighborhood of p as a disjoint union of parallel straight line segments, such that h is constant along each segment.
3. $p \in 2D$ if it is neither $0D$ or $1D$. Thus in a neighborhood of a $2D$ point the image function has variations along any direction.

The classification above produces a segmentation of the image into three sets of points: $0D$, $1D$ and $2D$. We will suppose that this segmentation is regular, by regularizing it if necessary.



(a) (b)

Figure 1: Perceptual segmentation.



(a)

Figure 2: Geometric segmentation.

According to the terminology introduced in [16], we will say that the segments $0D$, $1D$ and $2D$ have intrinsic dimensions 0, 1 and 2 respectively.

From a perceptual point of view, the segment $0D$ correspond to regions with small or no variation of the image intensity; the segment $1D$ correspond to neighborhoods of the edges and the segment $2D$ correspond to regions of high intensity variation in all directions. For this reason the segmentation introduced above is called a *perceptual segmentation*. Figure 1(b) shows a perceptual segmentation of the blurred image of a square shown in Figure 1(a).

How to devise good operators to compute the perceptual segmentation of an image? In order to cope with this question, we remark that the perceptual segmentation introduced above is closely related with the geometric classification of the points in the image surface as planar, parabolic, elliptic and hyperbolic. This classification gives the geometric segmentation that we will introduce in the next section.

3.2 Geometric Segmentation

A geometric segmentation of an image can be obtained using the geometric classification of points of the image surface. This segmentation is similar to the perceptual segmentation of the previous section, for this reason we will use the notation $g0D$, $g1D$ and $g2D$.

Definition (Geometric Segmentation). If $h\colon U \to \mathbb{R}$ is an image, with $U \subset \mathbb{R}^2$, then

1. $p \in g0D$, if the point $(p, h(p))$ is planar;

2. $p \in g1D$, if the point $(p, h(p))$ is parabolic;

3. $p \in g2D$, if the point $(p, h(p))$ is elliptic or hyperbolic.

The segmentation obtained using the above classification is called the *geometric segmentation* of the image.

The geometric segmentation has very interesting properties. It is possible to show, for example, that the common boundary between the regions $g0D$ and $g1D$ is a line segment, and the boundary between the region $g1D$ and $g2D$ is a differentiable path.

We will suppose that the geometric segmentation is regular (if not, we regularize it). Figure 2(a) shows a geometric segmentation of the same blurred square used in Figure 1(a). Note that for this image the perceptual segmentation coincides with the geometric segmentation.

In general, the geometric segmentation is similar to the perceptual segmentation. The $g0D$ regions consist of pieces of plane, the $g1D$ regions consist of pieces of cones or cylinders and the $g2D$ regions have a nonlinear behavior in all of the directions. In spite of the similarity, these two segmentations might not coincide exactly. In fact, pieces of cones could appear on $g2D$ regions. The following inclusions can be easily verified.

$$0D \subset g0D, \quad 1D \subset g0D \cup g1D,$$

$$2D \subset g1D \cup g2D, \quad g2D \subset 2D.$$

The similarity between the perceptual and the geometrical segmentations indicate to us that operators based on the curvature of the image surface could be used to detect and eliminate redundancies on an image. An interesting fact about the curvature is that it is a nonlinear point operation.

Invariance. We should remark that both the geometric and perceptual segmentations are invariant by rotations, translations, change in scale, and also by changes in the image brightness or contrast.

4 Reconstruction

In order to use curvature operators to detect and eliminate redundancy, it is very important that we are able to reconstruct the image from the transformed, non-redundant, image. Theorem 1 below shows that we can recover an image from the $2D$ points, as was suggested in [2, 16].

Consider a surface S with gaussian curvature $K = 0$ and denote the set of planar points by F. The set P of parabolic points will be given by $P = S - F$. Note that if p is a point of the surface and k_1 and k_2 are eigenvectors of the Gauss map, than $k_1(p) = k_2(p) = 0$ is a closed property. Therefore, the sets F and P are, respectively, closed and open in S.

Theorem 1. *A compact surface is completely determined by the points with non-zero Gaussian curvature.*

To prove the Theorem we need two Lemmas.

Lemma 1. *If S is a surface with zero Gaussian curvature, then the only asymptotic line that passes through a parabolic point $p \in P \subset S$ is an open line segment.*

Lemma 2. *Consider a surface S with zero Gaussian curvature. If s is the arc length of the asymptotic line r through a parabolic point $p \in P$, and $H(s)$ is the mean curvature along r, then*

$$H(s) = \frac{1}{as + b},$$

for some $a, b \in \mathbb{R}$.

The proofs of the above lemmas can be found on [5]. Now we will prove Theorem 1.

Proof of Theorem 1. Consider a compact surface S, and let R be the topological closure of the set of points with non-zero Gaussian curvature. Since S is compact, we have $R \neq \emptyset$. Denote by S' the complement of R in S, that is $S' = S - R$. Of course, S' is a surface with Gaussian curvature $K = 0$ everywhere. Now we will show that the direction of the eigenvectors of the Gauss map in the boundary of R determine the surface S outside of the region R.

If $p \in P \subset S'$ is a parabolic point, and r is the asymptotic line through p, from Lemma 1, r is a line segment. Since S is compact, r is bounded. Let p_1 and p_2 the boundary points of r. We can use Lemma 2 to conclude that the mean curvature of S satisfies

$$H(s) = \frac{1}{as + b},$$

along r. From the continuity of H we have $H(p_1) \neq 0$, and the same result applies to the point p_2. Therefore p_1 is not planar and since $K = 0$ is a closed property, we conclude that p_1 is parabolic.

From the maximality of r we conclude that p_1 belongs to the boundary $\partial S' = \partial R$. Since p_1 is not umbilic, because in this case it would be planar, the direction of the eigenvector of the derivative of the Gauss map corresponding to the eigenvalue 0 is uniquely determined, and coincides with the direction of r. This determines r uniquely for each parabolic point in $\partial S'$.

On the other hand, since the derivative of the Gauss map is identically zero in F, the connected components of F are constituted by pieces of planes and therefore they are completely determined by some point in ∂R, plus the normal vector at this point. This concludes the proof of the theorem.

It is interesting to note that the proof above is constructive, that is, it shows how we could devise an algorithm to reconstruct the surface from the points of nonzero curvature. For this, we just have to follow the lines of zero curvature, which are line segments, taking as the starting point the boundary of the set of points with nonzero curvature.

5 Edges

When we look at an image, it is possible to identify huge regions where we have a small variation of the image intensity. These regions appear separated by curves where great variations occur. These curves are called generically *edges*. Edges have a very important role in the perception of the image. In general, transformations that do not preserve the edges cause a noticeable degradation on the image quality.

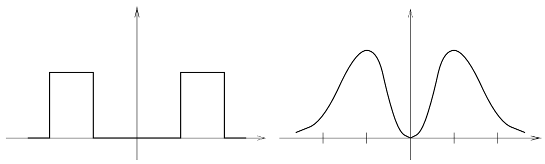
5.1 Edges and the Human Visual System

A commonly used technique in image processing consists in obtaining a segmentation $U = \bigcup U_\lambda$ in such a way that we do not have great variations of intensity in each segment U_λ. Since great variations occur on the neighborhood of the edges, the segmentation must be obtained in such a way that the edges are part of the boundaries of the segmentation.

Edge based segmentation have deserved special attention since David Marr conjectured that a complete set of edges provides a complete representation of the image. In this case, the edges would be computed from zero-crossings of the Laplacian

$$\Delta(h * g_\sigma) = h * \Delta g_\sigma,$$

where g_σ is the Gaussian with mean zero and variance σ. The rationale for using the filter Δg_σ comes from a Theorem by Loogan, [8], stating that if the ratio between the smallest and biggest frequencies is smaller than 1/2 (octave band), then the signal can be reconstructed, exactly, from the zero crossings.



(a) Octave band-pass ideal filter

(b) Laplacian of Gaussian

Figure 3: Band-pass filters on the frequency domain.

Note that Δg_σ is essentially a band-pass filter, even though it is not a octave band-pass filter (Figure 3). Certainly, if we can reconstruct $\Delta(h * g_\sigma)$, then $h * g_\sigma$ can be recovered because the equation

$$\Delta h = g$$

determines the function h up to a boundary condition.

Nevertheless, D. Marr calls the attention to the fact that perception involves more complex features. D. Marr's conjecture was proved by Hummel and Moniot, [7], with the additional hypothesis of having gradient information along the boundaries.

Recently, S. Mallat, [11], used dyadic wavelets to compute the multiscale edges, and also to reconstruct an image from its edges. Mallat's method can be interpreted as a sampling and reconstruction scheme where the sampling points are the edges and the samples are the derivatives of the image in a proper scale. In spite of the efforts of Mallat, Hummel and Moniot, Meyer, [13], has shown counter examples.

5.2 Edges and Gaussian Curvature

One of the problems with D. Marr's conjecture is related with the precise definition of edges. A common model consists in describing edges as the local maxima points of

$$\frac{dh}{dt} = |\nabla h|^2,$$

where $h(t)$ indicate the line integrals of ∇h.

Therefore, if $p \in U$ is an edge point and we use the notation $\nu = \nabla h(p)$, we have

$$\frac{\partial}{\partial \nu}\langle \nabla h, \nabla h \rangle = 0,$$

or equivalently,

$$\langle \operatorname{Hess} h \nabla h, \nabla h \rangle = 0.$$

Therefore, we have two possibilities

1. $\operatorname{Hess} h$ is singular in p;
2. ∇h and $\operatorname{Hess} h$ are orthogonal at p.

The second possibility implies that $\operatorname{Hess} h$ has the form

$$\operatorname{Hess} h = \begin{pmatrix} 0 & b \\ b & c \end{pmatrix}$$

with respect to an orthogonal basis with one of the vectors in the direction of $\nabla h(p)$. This is a rare possibility since these matrices constitute a set of measure zero on the set of symmetric matrices.

In sum, we conclude that in most of the cases the matrix $\operatorname{Hess} h$ is singular on the edge points of the image. Consequently, the boundary points have zero Gaussian curvature. Since p is an edge point, it does not belong to the interior of a planar region and, typically, it must belong to a line of planar or parabolic points.

The above analysis shows that typically the edge points belong to the lines of planar or parabolic points. This shows that the information contained in the parabolic points is extremely important for the reconstruction of the image, because of the sensitivity of the human visual system to errors involving edges. Also, this gives a cue to understand the cause of the distortions obtained when we try to reconstruct an image without taking into account the parabolic points. See, for example, the reconstructions in [2, 16].

Figure 4 illustrates the elliptic, hyperbolic and parabolic regions, and the union of the regions. Note how the combination of the elliptic, hyperbolic and parabolic regions can be combined to determine information on both sides of the edges of the figure.

6 Implementation

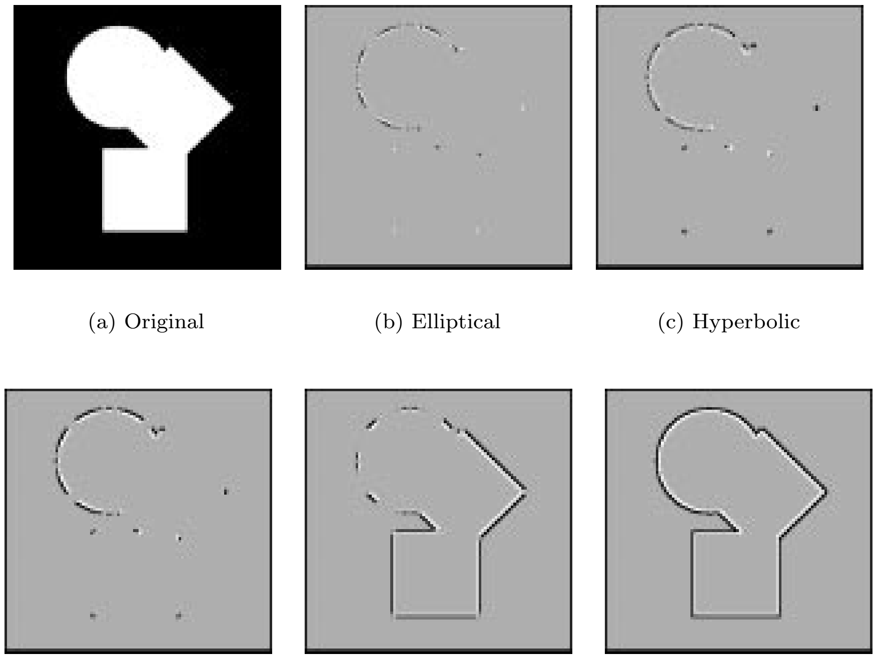
Theorem 1 shows that an image can be reconstructed from the partial knowledge of its values, if we preserve information on the points where the Gaussian curvature is non zero. Moreover, the theorem gives a process to reconstruct the surface. In practice however, this reconstruction process suffers from numerical errors.

The image can be easily reconstructed if, besides the knowledge of its values on the regions of nonzero Gaussian curvature ($g2D$), we also know the image on the parabolic regions ($g1D$). In this case, the missing points belong to pieces of planes and therefore we can use linear interpolation along the directions u and v to obtain

$$h_{ij} = \frac{1}{2}(h_{i-1j} + h_{i+1j}),$$

and

$$h_{ij} = \frac{1}{2}(h_{ij-1} + h_{ij+1}).$$


(a) Original (b) Elliptical (c) Hyperbolic

(d) Ellip + Hyper (e) Parabolic (f) All

Figure 4: Regions

From this we obtain

$$h_{i-1j} + h_{i+1j} + h_{ij-1} + h_{ij+1} - 4h_{ij} = 0. \qquad (5)$$

The above system is linear and its solution gives a complete reconstruction of the image. Figure 5 shows the reconstruction of the Lena image using this method:

1. (a) shows the original image;
2. (b) shows the parabolic points;
3. (c) shows the two-dimensional points (Elliptcals and Hyperbolics);
4. (d) shows the edges computed as the union of the points in (b) and (c);
5. (e) shows the reconstructed image.

Since some threshold is used in order to compute the segmentation, equation (5) actually reconstructs the missing parts of the Monge surface as hyperbolic patches.

7 Conclusions

In this paper we used differential geometry to segment and reconstruct an image from its bidimensional intrinsic characteristics. We presented a theoretical proof that it is possible to reconstruct a surface from its points of non-zero Gaussian curvature. This result answers in an affirmative way the question posed in [2, 16]. We also presented an algorithm that reconstructs the image from the points $g1D$ and $g2D$.

We analyzed the image edges from the point of view of differential geometry, and we have shown that typically, the edge points belong to parabolic regions of the image. This shows that we can introduce reconstruction errors in the neighborhood of the edges if we do not take into account the $1D$ regions. These errors are perceptually critical because of the sensitivity of the human visual system to these image features.

7.1 Future work

The results of this paper have a great potential for applications. Currently, we are investigating several applications of the technique as well as continuing with the theoretical studies. Some of the topics we are investigating are described below:

1. Obtain efficient reconstruction algorithms, and also investigate the importance of the parabolic points in the reconstruction process.

(a) Original

(b) 1D

(c) 2D

(d) 1D + 2D

(e) Reconstruction

Figure 5: Reconstruction.

2. Image compression is a natural arena to test these techniques. In fact the non-linear curvature operators could originate an edge base compression technique.

3. Curvature based filters can be use to obtain an adaptive representation of an image.

4. Use the curvature operator to compute a signature for the image. This signature could be used to develop a QBIC technique for image databases.

5. The concepts and techniques described in this paper extend to different graphical objects. We are currently working on a generalization to volumetric graphical objects. In this case the volumetric image is represented as a hypersurface in $\mathbb{R}^4$.

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