A Simple Algorithm for Decomposing Convex Structuring Elements

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Abstract. A finite subset of $\mathbb{Z}^2$ is called a structuring element. This paper presents a new and simple algorithm for decomposing a convex structuring element as sequence of Minkowski additions of a minimum number of subsets of the elementary square (i.e., the 3×3 square centered at the origin). Besides its simplicity, the advantage of this algorithm over some known algorithms is that it generates a sequence of non necessarily convex subsets, what means subsets with smaller cardinality and, consequently, faster implementation of the corresponding dilations and erosions. The algorithm is based on algebraic and geometrical properties of Minkowski additions. Theoretical analysis of correctness and computational time complexity are also presented.

keywords: convex structuring element, decomposition, Minkowski addition.

1 Introduction

A finite subset $\mathbb{Z}^2$ is called a structuring element (SE). In this paper, we consider just non empty SE's. The problem of decomposing a SE as a sequence of Minkowski additions of smaller subsets has been studied by several researchers [15, 11, 13, 7, 9, 10, 14, 2, 3, 1] and many different algorithms have arisen to generate decompositions. In fact, this problem is very hard, since not all SE's have sequential decompositions by Minkowski additions [13]. In addition, it is not known an efficient algorithm for determining the existence of such decompositions for an arbitrary SE.

However, it is well known that all convex SE has sequential decompositions [13, 9]. Xu [13] developed a complex algorithm for the decomposition of convex SE's in terms of a minimum number of small convex subsets In fact, Xu's algorithm can be divided into two steps: (i) compute a decomposition of a convex SE by solving a system of linear equalities with a fixed number of variables; (ii) apply an intricate optimization process, subdivided in many complex cases, to join, by Minkowski addition, the subsets found in (i). Park and Chin [9] developed an extension of step (i) of Xu's algorithm and provided an algorithm for finding an optimum decomposition of convex SE's for 4-connected parallel array processors. In both algorithms, all elements of the decomposition are convex subsets of the elementary square (i.e., the 3×3 square centered at the origin).

Here, we present a very simple algorithm for the generation of decompositions of a convex SE as a sequence of Minkowski additions of a minimum number of subsets (non necessarily convex) of the elementary square. The computational time complexity of algorithms that implement erosion and dilation depend on the number of points of the SE, thus, our algorithm has an advantage over Xu's and Park and Chin's algorithms, since all elements in the output of their algorithms contain subsets generated in our decomposition. Another advantage is that our algorithm can be easily understood and implemented.

Following this introduction, Section 2 presents the mathematical foundations necessary for presenting the algorithm. Section 3 gives the proposed algorithm. Sections 4 and 5 present, respectively, the proof of correctness and the time complexity analysis. Finally, Section 6 gives some conclusions and future steps of this research.

2 Mathematical Foundations

This section gives the mathematical foundations necessary for presenting the proposed decomposition algorithm. Subsection 2.1 states the problem of decomposition of SE's in terms of a sequence of Minkowski additions. Subsection 2.2 presents definitions and properties used in the decomposition algorithm.

2.1 Problem Statement

For any SE X and $y \in \mathbb{Z}^2$, X_y denotes the *translation* of X by y, that is, $X_y = \{x \in \mathbb{Z}^2 : x - y \in X\}$.

Let X and Y be SE's. The *Minkowski addition* and *subtraction* of X and Y are the SE's given, respectively, by $X \oplus Y = \cup\{X_y : y \in Y\}$ and $X \ominus Y =$

Figure 1: (a) A SE A. (b) The convex hull of A.

$\cap\{X_{-y} : y \in Y\}$.

We take the point $o = (0,0)$ as the *origin* of $\mathbb{Z}^2$. We call the 3×3 square centered at the origin the *elementary square*.

The *dilation* and the *erosion* by the SE A are the mappings given, respectively, by, for any $X \subseteq \mathbb{Z}^2$, $\delta_A(X) = X \oplus A$ and $\epsilon_A(X) = X \ominus A$.

A property of dilations and erosions is their sequential decomposability [12, p. 47].

Proposition 1 *Let $A, B_1, B_2, \cdots, B_n$ be SE's. $\delta_A = \delta_{B_1}\delta_{B_2}\cdots\ \delta_{B_n}$ and $\epsilon_A = \epsilon_{B_1}\epsilon_{B_2}\cdots\ \epsilon_{B_n}$ if and only if $A = B_1 \oplus B_2 \oplus\ \cdots\ \oplus\ B_n$.*

The speed up achieved by representing erosions and dilations by a SE decomposed as a sequence of Minkowski additions in terms of subsets of the elementary square, in conventional machines, was quantitatively studied by Maragos [8, p. 77], who showed examples where the time complexity of the algorithms that implement erosions and dilations went from quadratic, in the direct implementation, to linear, in the decomposition of the SE by Minkowski additions.

Given a SE A, a *sequence of subsets of A* is the succession of subsets of A in a fixed order. For example, if $B_1, B_2, B_3, B_4, B_5, B_6, B_7$ are distinct subsets of A, then $[B_7, B_1, B_1, B_1, B_2, B_2, B_3, B_1,\ B_4, B_5,\ B_2,\ B_6]$ is a sequence of subsets of A. In this paper, we consider just finite sequences.

Given a SE A and two sequences of subsets of A, say S and R, the *concatenation of the sequences* S and R, denoted by $S \cdot R$, is the sequence formed by the elements of S followed by the elements of R. For example, if $S = [B_1, B_2]$ and $R = [B_1, B_3]$ are two sequences of subsets of A, then the concatenation $T = S \cdot R$ is the sequence $T = [B_1, B_2, B_1, B_3]$.

A SE A is said to have a *sequential decomposition* (or A is *decomposable*) if there exists a sequence $[B_1, \cdots, B_n]$ of subsets of the elementary square such that $A = B_1 \oplus B_2 \oplus \cdots \oplus B_n$. The sequence $[B_1, B_2, \cdots, B_n]$ is called a *decomposition sequence* of A.

A decomposition sequence of a SE can be decomposed in two subsequences: *shape* and *translation*. The shape subsequence represents the shape of the SE and it is formed by the subsets in the sequence that have at least two points. The translation subsequence defines the position of the SE in the integer plane and it is formed by the unitary subsets in the sequence. The shape subsequence $[B_1, \cdots, B_k]$ is called, respectively, the *shape decomposition* (or simply, *decomposition*) of A and the number k is the *length* of the decomposition of A.

The convex hull $C(A)$ of a SE A is the intersection of all half planes that contain A. We suppose that subsets are represented in a square grid and consider just the half planes with slopes 0, 45, 90 and 135 degrees to build the convex hull (see Figure 1 for an example). A SE A is said *convex* if and only if $A = C(A)$.

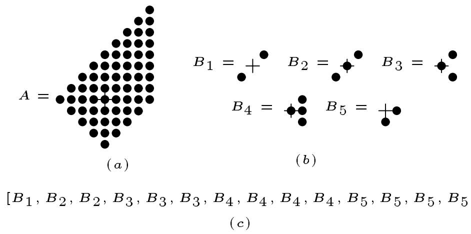

Figure 2: (a) A SE A. (b) The subsets of the elementary square that are in $\mathcal{B}(A)$. (c) The invariant sequence of A.

In this paper, we are interested in solving the following problem.

Problem 1 *Given a convex SE A, find a shape decomposition of A with minimum length.*

Xu [13] presented an intricate algorithm for solving this problem and stated the following proposition.

Proposition 2 *If A is a convex SE, then A has a sequential decomposition.*

In this paper, we view the problem with a different optics and present a very simple algorithm for decomposing convex SE's.

2.2 Definitions and Properties for the Algorithm

Let A and B be SE's. We say that B is an *invariant SE* (or simply, an *invariant*) of A if and only if $A = (A \ominus B) \oplus B$.

Example 1 *The SE's B_1, B_2, B_3, B_4 and B_5, presented in Figure 2b, are invariants of the SE A, presented in Figure 2a.*

Propositions 3 and 4 give some properties of invariants of a given SE. The first one was stated by Serra [12, p. 53] and the second one by Zhuang and Haralick [15, Proposition 5].

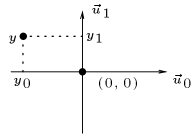

Figure 3: Axis $\vec{u}_0$ and $\vec{u}_1$.

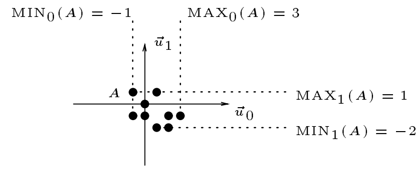

Figure 4: A SE A with the axis $\vec{u}_0$ and $\vec{u}_1$.

Proposition 3 *Let A and X be SE's. X is invariant of A if and only if there exists a SE Y such that $A = Y \oplus X$.*

Proposition 4 *Let A, X and Y be SE's. If $A = X \oplus Y$, then X and Y are both invariants of A.*

The following corollary is an immediate consequence of the Proposition 4 and was stated in [5].

Corollary 5 *Let A be a SE. If $[B_1, B_2, \cdots, B_k]$ is a shape decomposition of A, then each B_i is an invariant of A.*

Let X be a SE and let n be a positive integer. The succession of $n-1$ Minkowski additions of X is the SE denoted by nX, that is, $nX = ((X \oplus X) \oplus \cdots \oplus X)$. This notation is extrapolated for $n = 0$ by stating $0B = \{(0,0)\}$.

Let A and X be SE's such that X is an invariant of A. The *multiplicity* of X with respect to A is the greatest positive integer n such that nX is an invariant of A.

Example 2 *The multiplicity of the SE's B_1, B_2, B_3, B_4 and B_5, presented in Figure 2b, with respect to A, presented in Figure 2a, are, respectively, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$. $n_4 = 4$ and $n_5 = 4$, since, for any $i \in \{1, 2, 3, 4, 5\}$, $(n_i + 1)B_i$ is not an invariant of A.*

Let us state an equivalence relation on a generic collection $\mathcal{C}$ of subsets of $\mathbb{Z}^2$. Let X and Y be two elements of $\mathcal{C}$. We say X and Y are *equivalent under translation* if and only if one can be built by a translation of the other, that is, $X \equiv Y$ if and only if there exists $h \in \mathbb{Z}^2$ such that $X_h = Y$.

Since the equivalence under translation is an equivalence relation (i.e., reflexive, symmetric and transitive), the set of their equivalence classes (i.e., the sets composed exactly by all the equivalent elements in $\mathcal{C}$) constitutes a partition of $\mathcal{C}$.

We denote by $\mathbf{P}(\mathcal{C})$ the set of all the equivalence classes (under translation) on $\mathcal{C}$. We denote by $\mathbf{E}(\mathcal{C})$ a set composed by exactly one element of each equivalence class in $\mathbf{P}(\mathcal{C})$, that is, $\mathbf{E}(\mathcal{C})$ is a set such that $|\mathbf{E}(\mathcal{C})| = |\mathbf{P}(\mathcal{C})|$.

The set of all subsets of the elementary square that have at least two points is denoted $\mathcal{Q}$, that is, $\mathcal{Q} = \{B \subseteq \{-1, 0, 1\}^2 : |B| \geq 2\}$.

Given a SE A, the set of all elements of $\mathbf{E}(\mathcal{Q})$ that are invariant of A is denoted $\mathcal{B}(A)$, that is, $\mathcal{B}(A) = \{B \in \mathbf{E}(\mathcal{Q}) : B \text{ is an invariant of } A\}$.

Example 3 *The set $\mathcal{B}(A)$ for the SE A presented in Figure 2a is $\mathcal{B}(A) = \{B_1, B_2, B_3, B_4, B_5\}$, where B_1, B_2, B_3, B_4 and B_5 are the SE's presented in Figure 2b.*

The next proposition was stated in [5] and it gives an important property of multiplicity of a SE.

Proposition 6 *Let A be a SE and $X \in \mathcal{B}(A)$. If n is the multiplicity of X with respect to A, then any decomposition sequence of A contains at most n elements equal to X.*

Let X be a SE and n be a non-negative integer. If $n \neq 0$, then the sequence formed by the succession of n subsets X is denoted by $\mathsf{Seq}[X, n]$, that is, $\mathsf{Seq}[X, n] = [X, X, \cdots, X]$. If $n = 0$, $\mathsf{Seq}[X, 0]$ denotes the empty sequence.

Let A be a SE. Let $B_1, B_2, \cdots, B_k$ be all elements of $\mathcal{B}(A)$ in a fixed order and n_i be the multiplicity of B_i with respect to A ($i = 1, \cdots, k$). The *invariant sequence* of A is the sequence $\mathsf{SeqInv}[A] = \mathsf{Seq}[B_1, n_1] \cdot \mathsf{Seq}[B_2, n_2] \cdot \cdots \cdot \mathsf{Seq}[B_k, n_k]$.

Example 4 *The sequence $[B_1, B_2, B_2, B_3, B_3, B_3, B_3, B_4, B_4, B_4, B_4, B_5, B_5, B_5, B_5]$ (presented in Figure 2c) is the invariant sequence of the SE A presented in Figure 2a.*

Now, we will state a lower bound for shape decompositions of a given SE. For that, we analyze some measures taken over the SE's.

Let $\vec{u}_0$ and $\vec{u}_1$ be the Cartesian axis that intersect the origin and have slopes, respectively, 0 and 90 degrees (see Figure 3). For a given point $x \in \mathbb{Z}^2$, let x_0 and x_1 be the orthogonal projections of x at the Cartesian axis $\vec{u}_0$ and $\vec{u}_1$, respectively.

Example 5 *The orthogonal projections of the point $y = (-5, 2) \in \mathbb{Z}^2$ are $y_0 = -5$ and $y_1 = 2$ (see Figure 3).*

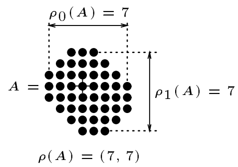

Figure 5: Rectangular projection of A.

Let A be a SE. For $i = 0, 1$, let $\mathrm{MAX}_i(A)$ and $\mathrm{MIN}_i(A)$ be, respectively, the maximum and the minimum orthogonal projections at the Cartesian axis $\vec{u}_i$ of the points in A, that is, $\mathrm{MAX}_i(A) = \max\{x_i : x \in A\}$ and $\mathrm{MIN}_i(A) = \min\{x_i : x \in A\}$.

Example 6 *For example, the maximum and the minimum orthogonal projections of the SE A presented in Figure 4 are, respectively,* $\mathrm{MAX}_0(A) = 3$, $\mathrm{MAX}_1(A) = 1$ *and* $\mathrm{MIN}_0(A) = -1$, $\mathrm{MIN}_1(A) = -2$.

The next proposition, stated in [5], gives a property of maximum and the minimum orthogonal projections.

Proposition 7 *If A and B are SE's, then* $\mathrm{MAX}_i(A \oplus B) = \mathrm{MAX}_i(A) + \mathrm{MAX}_i(B)$ *and* $\mathrm{MIN}_i(A \oplus B) = \mathrm{MIN}_i(A) + \mathrm{MIN}_i(B)$.

The *rectangular projection* of the SE A is the vector $\rho(A) \in \mathbb{Z}^2$ such that, for $i \in \{0, 1\}$, $\rho_i(A) = \mathrm{MAX}_i(A) - \mathrm{MIN}_i(A)$. In other words, the coordinates of the rectangular projection of a SE A are the lengths of the edges of the smallest rectangle that contains A. Note that the rectangular projection is independent of translation, that is, $\rho(A) = \rho(A_h)$, for any $h \in \mathbb{Z}^2$.

Example 7 *The rectangular projection of the subset A presented in Figure 5 is $\rho(A) = (7, 7)$.*

One interesting property of this measure is given in the following proposition and was stated in [5].

Proposition 8 *Let A, X, Y be SE's. If $A = X \oplus Y$, then $\rho(A) = \rho(X) + \rho(Y)$.*

The next proposition, stated in [5], gives a lower bound for the length of decompositions of a given SE.

Proposition 9 *Let A be a SE. If A has a decomposition, then a shape decomposition of A contains at least $lower(A) = \lceil \max\{\rho_0(A),\, \rho_1(A)\}/2 \rceil$ elements.*

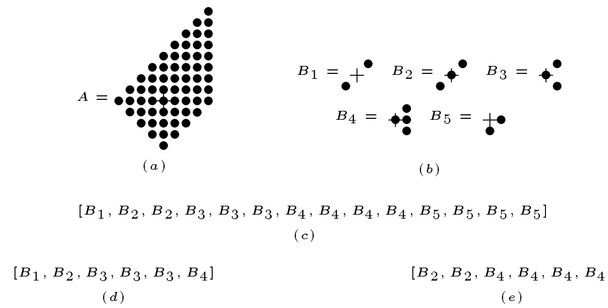

Figure 6: (a) A SE A. (b) The subsets of the elementary square that are in $\mathcal{B}(A)$. (c) The invariant sequence of A according to the order chosen to construct it. (d) Output of algorithm Decompose. (e) Output of Xu's algorithm.

3 Search of an Optimum Decomposition

This section presents the proposed algorithm for the decomposition of convex SE's.

Algorithm Decompose:

Input: A convex SE A that is not an unitary set.
Output: Optimum shape decomposition of A.

01: Let $[B_1, B_2, \cdots, B_n]$ be the invariant sequence of A.
02: $Y_0 \leftarrow \{(0,0)\}$
03: $i \leftarrow j \leftarrow 0$
04: $\mathsf{S}_0(A) \leftarrow [\,]$ /* empty sequence */
05: **while** $(\rho(Y_j) \neq \rho(A))$ **do**
06:   **begin**
07:     $i \leftarrow i + 1$
08:     **if** ($Y_j \oplus B_i$ is an invariant of A) **then**
09:       **begin**
10:         $j \leftarrow j + 1$
11:         $\mathsf{S}_j(A) \leftarrow \mathsf{S}_{j-1}(A) \cdot [B_i]$
12:         $Y_j \leftarrow Y_{j-1} \oplus B_i$
13:       **end**
14:   **end**
15: output the sequence $\mathsf{S}_j(A)$.

Depending on the order chosen to construct the invariant sequence at Step 01 in Algorithm Decompose, different outputs (shape decompositions) arise. We have sorted the elements of the invariant sequence in decreasing order, according to the sum of the coordinates of the rectangular projections of each subset in the invariant sequence, and, at the same time, in increasing order, according to the number of points of each subset in the invariant sequence. For example, Figure 6c presents the invariant sequence of the SE A (presented in Figure 6a) according to the order chosen to construct it. In this figure, observe that

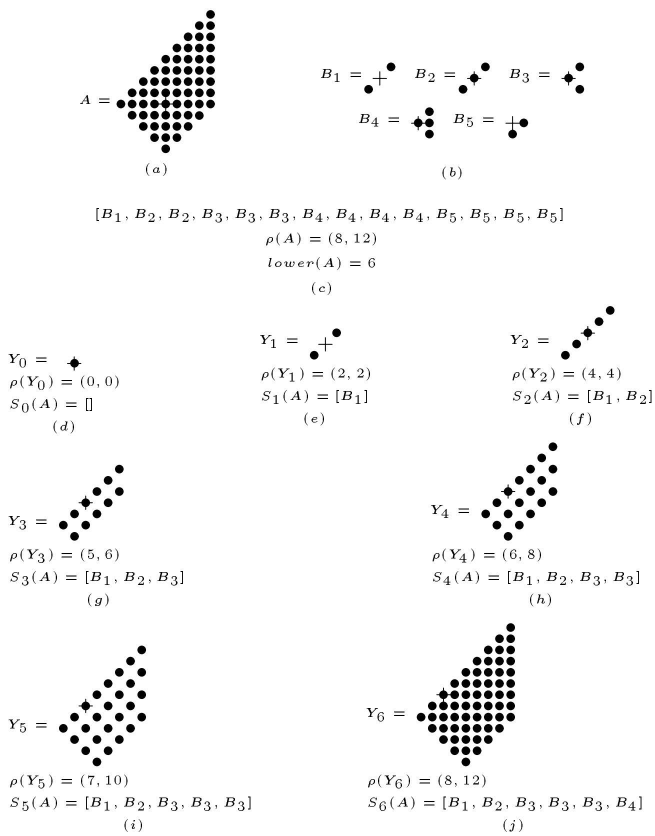

Figure 7: An example showing the algorithm running.

$\rho_0(B_1)+\rho_1(B_1) = \rho_0(B_2)+\rho_1(B_2) > \rho_0(B_3)+\rho_1(B_3)$ and B_1 contains less points than B_2.

According to this sorting, the algorithm DECOMPOSE prefers to choose non-convex SE's rather than convex ones for the shape decomposition. Thus, as the time complexity of algorithms that implement erosions and dilations depends on the number of points in the SE, the algorithm DECOMPOSE has an advantage over Xu's algorithm, since all elements in the output of Xu's algorithm are convex subsets of the elementary square [13]. For an example, in Figures 6d and 6e are presented, respectively, the output of our and Xu's algorithm. In this particular example, the difference is just four points, but for bigger SE's the difference can be considerable.

In Figure 7, we show a simple example of the algorithm DECOMPOSE running for finding a shape decomposition of the convex SE A, presented in Figure 7a. The invariant sequence, the rectangular projection and the lower bound of A are presented in Figure 7c. Figure 7d presents the unitary set Y_0 that contains the origin and the empty sequence $S_0(A)$ that are computed, respectively, at steps 02 and 04. Figure 7e presents the set $Y_1 = Y_0 \oplus B_1$ (computed at step 12) and the sequence $S_1(A) = [B_1]$ (computed at step 11), since $Y_0 \oplus B_1$ is an invariant of A (checked at step 08). In a similar way, in Figure 7f, we show the set $Y_2 = Y_1 \oplus B_2$ (computed at step 12) and the sequence $S_2(A) = [B_1, B_2]$ (computed at step 11), since $Y_1 \oplus B_2$ is an invariant of A (verified at step 08). The next subset B_3 is not selected to be concatenated with $S_2(A)$, since $Y_2 \oplus B_3$ is not an invariant of A (checked at step 08). Figures 7g, 7h and 7i present the sets $Y_3 = Y_2 \oplus B_3$, $Y_4 = Y_3 \oplus B_3$ and $Y_5 = Y_4 \oplus B_3$ (computed at step 12) and the sequences $S_3(A)$, $S_4(A)$ and $S_5(A)$ (computed at step 11), since $Y_2 \oplus B_3$, $Y_3 \oplus B_3$ and $Y_4 \oplus B_3$ are invariant of A (verified at step 08). In Figure 7j, the set $Y_6 = Y_5 \oplus B_4$ and the sequence $S_6(A)$ are computed (at steps, respectively, 12 and 11), since $Y_5 \oplus B_4$ is an invariant of A. In addition, since $\rho(Y_6) = \rho(A)$, then the algorithm stops. Observe that $Y_6 \equiv A$ and $lower(A) = 6$. So, $S_6(A)$ is an optimum decomposition for the convex SE A.

4 Correctness of the Algorithm

In this section, we prove the correctness of the algorithm. Although the algorithm DECOMPOSE is simple, its correctness proof is not intuitive. We divide this section in two parts. In the first, we prove that the algorithm output is a shape decomposition of a given convex SE and in the last one, we show that this decomposition is optimum.

4.1 The Output of the Algorithm is a Shape Decomposition

Given a convex SE A. Let $\mathsf{SeqInv}[A] = [B_1, B_2, B_3, \cdots, B_n]$. At step 12, the algorithm computes the SE $Y_j = Y_{j-1} \oplus B_i$. For each j computed by the algorithm, we call the *invariant built sequence* of A the sequence $[Y_0, Y_1, Y_2, \cdots, Y_j]$ and the *invariant remainder sequence* of A the subsequence $\mathsf{Remainder}[Y_j] = [B_{i+1}, B_{i+2}, \cdots, B_n]$ of $\mathsf{SeqInv}[A]$.

Given a convex SE A, we will denote $\mathsf{S}_j(A) = [C_1, C_2, \cdots, C_j]$ the sequence built by the algorithm when it computes the SE $Y_j = Y_{j-1} \oplus C_j$. Note that, if $[Y_0, Y_1, Y_2, \cdots, Y_j]$ is an invariant built sequence of A, then, C_j is the first element in $\mathsf{Remainder}[Y_{j-1}]$ such that $Y_{j-1} \oplus C_j$ is an invariant of A.

Given a convex SE A, if the invariant sequence of A is built in the manner described in Section 3, then the following proposition, stated in [4], gives an important result in order to prove that the output of the algorithm is a shape decomposition of A.

Proposition 10 *Let A be a convex SE. If the sequence*

$[Y_0, Y_1, Y_2, \cdots, Y_j]$ *is the maximal invariant built sequence of* A*, then* $\rho(Y_j) = \rho(A)$ *and* $Y_j \equiv A$.

Given a convex SE A, when a maximal invariant built sequence $[Y_0, Y_1, Y_2, \cdots, Y_j]$ is found, then, by Proposition 10, the rectangular projections of A and Y_j are equal and, therefore, the algorithm stops (step 05), and also by Proposition 10, $Y_j \equiv A$ and therefore, the sequence $\mathsf{S}_j(A) = [C_1, C_2, \cdots, C_j]$ is a decomposition of A. It remains to show that the length of $\mathsf{S}_j(A)$ is minimum.

4.2 The Output of the Algorithm is an Optimum Shape Decomposition

The next proposition, stated in [4], is necessary to prove that the length of $\mathsf{S}_j(A)$ is minimum.

Proposition 11 *Let* A *be a convex SE. If the sequence* $[Y_0, Y_1, Y_2, \cdots, Y_j]$ *is an invariant built sequence of* A*, then* $lower(Y_j) = j$.

Finally, the following theorem gives the proof that the output of the algorithm is an optimum shape decomposition of a given convex SE A.

Theorem 12 *Let* A *be a convex SE. If the sequence* $[Y_0, Y_1, Y_2, \cdots, Y_j]$ *is the maximal invariant built sequence of* A*, then* $\mathsf{S}_j(A)$ *is the optimum shape decomposition of* A.

Proof: By Proposition 10, $Y_j \equiv A$. So, $\rho(Y_j) = \rho(A)$, and, consequently, $lower(Y_j) = lower(A)$. By Proposition 11, $lower(Y_j) = j$. Since $\mathsf{S}_j(A)$ contains exactly j elements and $j = lower(A)$, then, clearly, $\mathsf{S}_j(A)$ is the optimum shape decomposition of A. ■

Given a convex SE A, when a maximal invariant built sequence $\mathsf{S}_j(A) = [Y_0, Y_1, \cdots, Y_j]$ is found, then, by Proposition 10, $Y_j \equiv A$ and therefore the sequence $\mathsf{S}_j(A) = [C_1, C_2, \cdots, C_j]$ is a decomposition of A. By Theorem 12, $\mathsf{S}_j(A)$ is the optimum shape decomposition of A.

5 Time Complexity of the Algorithm

In this section, we discuss about the time complexity of the algorithm DECOMPOSE. The crucial steps for the time complexity of the algorithm are the steps 01, 05 and 08. In this section, given a convex SE A, m_p and m_s denote, respectively, the number of points of A and the number of elements in the invariant sequence of A. Let $n = \rho_0(A) + \rho_1(A)$. Clearly, $m_p = O(n^2)$.

Subsections 5.1 to 5.3 present the time complexity analysis of steps 01, 05 and 08. Subsection 5.4 gives the time complexity of the algorithm.

5.1 Time Complexity of Step 01

In order to construct the invariant sequence of a given SE A, it is necessary to verify if each element $B \in \mathbf{E}(\mathcal{Q})$ is an invariant of A and compute the multiplicity of B with respect to A. The time complexity for verifying if $(A \ominus B) \oplus B = A$ is $O(m_p)$, since the number of points in B is at most 9. The multiplicity of a given SE with respect to A is at most $\max\{\rho_0(A), \rho_1(A)\} = O(n)$. The cardinality of $\mathbf{E}(\mathcal{Q})$ is at most 2^9 subsets of the elementary square. Thus, the time complexity to compute the invariant sequence of A is $O(m_p) \cdot O(n) \cdot 2^9$, that is, $O(n^3)$, since $m_p = O(n^2)$.

5.2 Time Complexity of Step 05

In order to verify if $\rho(Y_j) \neq \rho(A)$, we have to compute $\rho(Y_j)$ and $\rho(A)$.

Let $\mathsf{S}_j(A) = [C_1, C_2, \cdots, C_j]$ be the sequence constructed by the algorithm when it has computed the SE Y_j. The rectangular projections of each $B \in \mathbf{E}(\mathcal{Q})$ can be computed previously, since all elements in $\mathbf{E}(\mathcal{Q})$ are fixed. So, by Proposition 8, $\rho(Y_j) = \rho(Y_{j-1}) + \rho(C_j)$. We can consider that the rectangular projection $\rho(Y_{j-1})$ has already been computed. So, in this case, the time complexity for computing $\rho(Y_j)$ is $O(1)$.

We can compute $\rho(A)$ only once and compare it with $\rho(Y_j)$, for each j. The time complexity to compute $\rho(A)$ is $O(n)$ [5].

Once $\rho(Y_j)$ and $\rho(A)$ have been computed, the time complexity for comparing them is $O(1)$.

5.3 Time Complexity of Step 08

In order to verify if $Y_j \oplus B_i$ is an invariant of A, we have to check if $A = (\cdots((A \ominus C_1) \ominus C_2) \ominus \cdots \ominus C_j) \ominus B_i) \oplus C_1) \oplus C_2) \oplus \cdots \oplus C_j) \oplus B_i)$. The time complexity for computing, for $k = 1, 2, \cdots, j$, $A \ominus C_k$, $A \oplus C_k$, $A \ominus B_i$ or $A \ominus B_i$ is linear with respect to the number of points in A, that is, $O(m_p)$, since each C_k and B_i contain at most 9 points. Then, the time complexity for computing $A \ominus C_k$, $A \oplus C_k$, $A \ominus B_i$ or $A \ominus B_i$ is $O(n^2)$, since $m_p = O(n^2)$. So, the overall complexity for verifying if $Y_j \oplus B_i$ is an invariant of A is $O(j \cdot n^2)$.

5.4 Overall Time Complexity of the Algorithm

The multiplicity of a given SE with respect to A is at most $\max\{\rho_0(A), \rho_1(A)\}$ and the number of all possible subsets of the elementary square is 2^9. Thus, the number of elements m_s in $\mathsf{SeqInv}[A]$ is at most $2^9 \cdot \max\{\rho_0(A), \rho_1(A)\} = O(n)$, that is, $m_s = O(n)$.

By Proposition 11, $j = lower(Y_j)$, and, by Proposition 9, $lower(Y_j) = \lceil \max\{\rho_0(Y_j),\ \rho_1(Y_j)\}/2 \rceil$ and

$lower(A) = \lceil \max\{\rho_0(A), \rho_1(A)\}/2 \rceil = O(n)$. Then, $j = lower(Y_j) \leq lower(A) = O(n)$, since $\rho_0(Y_j) \leq \rho_0(A)$ and $\rho_1(Y_j) \leq \rho_1(A)$.

As each element of the invariant sequence of A is checked at most once in the loop, the number of executions of the comparison at steps 05 and 08 is at most $m_s = O(n)$.

Since each comparison at step 05 takes time $O(1)$, the overall time complexity of step 05 is $m_s = O(n)$.

Since each comparison at step 08 is $O(j \cdot n^2)$, for $j = 1, 2, \cdots, lower(A)$, the overall time complexity of step 08 is $O(1 \cdot n^2) + O(2 \cdot n^2) + \cdots + O(lower(A) \cdot n^2)$.

Hence, the overall time complexity of the algorithm for finding an optimum decomposition of a convex SE A is $O(lower(A)^2 \cdot n^2)$, that is, $O(n^4)$, since $lower(A) = O(n)$.

6 Conclusion

In this paper, we present a new and simple algorithm for sequential decomposition of a convex SE in terms of a minimum number of subsets (not necessarily convex) of the elementary square. The time complexity of this algorithm is $O(n^4)$, where n is the sum of the coordinates of the rectangular projection of the input convex SE.

Although the time complexity of algorithms of Xu [13] and Park and Chin [9] is $O(n^2)$, the first one is very complicated and the second one is an extension of Xu's algorithm for $4-$connected parallel array processors.

All elements in the decomposition given by these algorithms are convex. So, as the computational time complexity of algorithms that implement erosion and dilation depends on the number of points in the SE, our algorithm has an advantage over Xu's and Park and Chin's algorithms.

Finally, theoretical analysis (proof of correctness and computational time complexity) of the proposed algorithm is presented.

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