An Optimal Algorithm to Construct All Voronoi Diagrams for k Nearest Neighbor Search in $\mathbb{T}^2$

PEDRO J. DE REZENDE [1] RODRIGO B. WESTRUPP [1]

[1] Instituto de Computação - Universidade Estadual de Campinas

Abstract. In this paper, we generalize to the oriented projective plane $\mathbb{T}^2$ an algorithm for constructing all order k Voronoi diagrams in the Euclidean plane. We also show that, for fixed k and for a finite set of sites, an order k Voronoi diagram in $\mathbb{T}^2$ has an exact number of regions. Furthermore, we show that the order k Voronoi diagram of a set of n sites in $\mathbb{T}^2$ is antipodal to its order $n-k$ Voronoi diagram, $\forall k:\ 1 \leq k < n$.

1 Introduction

The problem of answering k nearest neighbor queries with arbitrary k has an efficient solution based on the a priori construction of all order k Voronoi diagrams. This is the approach used in the optimal algorithm due to Frank Dehne [1] for the Euclidian Plane. We present a generalization of this algorithm for the two-dimensional oriented projective space $\mathbb{T}^2$ [8]. This space handles oriented lines as well as many other fundamental geometric concepts in a consistent way. Section 2 gives some references related to Oriented Projective Geometry. In section 3, we present several properties of Voronoi diagrams, some of them intrinsic of this space. For example, the order k Voronoi diagram of a finite set of sites in $\mathbb{T}^2$ has an exact number of regions. Furthermore, this diagram is antipodal to the order $n-k$ Voronoi diagram of the same set of sites, $\forall k:\ 1 \leq k < n$. Finally, section 4 presents the algorithm and a proof of its correctness.

2 Oriented Projective Geometry

For the sake of conciseness and due to limited availability of space, we refer the reader who might be unfamiliar with the basic concepts and notations related to Oriented Projective Geometry to the books [7] and [8]. It might also be of interest to read the theses [2], [6] and the papers [3], [5].

3 Voronoi Diagrams in $\mathbb{T}^2$

In this section, we present several properties of Voronoi diagrams in $\mathbb{T}^2$. Some properties are general and can be applied to Voronoi diagrams in $\mathbb{R}^2$, while others are intrinsic of $\mathbb{T}^2$.

3.1 Properties of Order k Voronoi Diagrams in $\mathbb{T}^2$

Definition 1: Let $p,\, q \in \mathbb{T}^2$, where at least one of them is a proper point. $d_{\mathbb{T}^2}(p,\, q)\ :\ \mathbb{T}^2 \times \mathbb{T}^2\ \rightarrow\ \mathbb{T}^1$, is defined as:

$$d_{\mathbb{T}^2}(p,q) = \left[\sqrt{(p_x q_w - q_x p_w)^2 + (p_y q_w - q_y p_w)^2}, p_w q_w\right]$$

Definition 2: The order relation between $\mathbb{T}^1$ points is defined as follows. Let $p,\, q$ be points in $\mathbb{T}^1$. We say that $p \leq_{\mathbb{T}^1} q$ if and only if:

$(i)\ p_w q_w \leq 0,\ p_w \geq 0,\ q_w \leq 0\ \ or$
$(ii)\ p_w q_w > 0,\ p_x q_w \leq q_x p_w.$

It is easy to verify that $d_{\mathbb{T}^2}$ satisfies the same axioms as (real-valued) distance functions and it is, therefore, a generalization of this notion. We call $d_{\mathbb{T}^2}(p,\, q)$ the *two-sided Euclidean Plane distance* from p to q.

In the following sections, we let $S \subset \mathbb{T}^2$ denote a set of n points, called *sites*. Let $\neg S$ denote the set of points that are antipodal to the sites in S. Let k be an integer, $1 \leq k < n$. For each subset $H_k \subset S$ of k sites, denote by $\mathcal{V}_k(H_k,\, S)$ the set:

$$\{x \in \mathbb{T}^2 \mid d_{\mathbb{T}^2}(x,\, p) \leq_{\mathbb{T}^1} d_{\mathbb{T}^2}(x,\, q),\ \forall p \in H_k,\ \forall q \subset S \setminus H_k\},$$

called the order k Voronoi region associated to H_k. That is, $\mathcal{V}_k(H_k, S)$ is the set of points in $\mathbb{T}^2$ whose k nearest neighbors in S are the sites in H_k. It follows from $d_{\mathbb{T}^2}$ being a generalization of the Euclidean metric that the set of all order k Voronoi regions forms a partition of $\mathbb{T}^2$. We denote this partition $\mathbb{V}_k(S)$, and call it the order k *Voronoi diagram* of S.

We assume that $S \cup \neg S$ contains no four points with proper circumcenter. Under this assumption, we show that each Voronoi vertex has degree 3, even if it lies on the line at infinity Ω.

Every edge of an order k Voronoi diagram is a portion of a bisector $B(p,\, q)$, with $p,\, q \in S$. We denote by $\bar{B}_k(p,\, q)$ the portion of $B(p,\, q)$ which is an edge of the order k Voronoi diagram of S, denoted by $\mathbb{V}_k(S)$. If $H \subseteq S$ and $|H| = k$, $\mathcal{V}_k(H,\, S)$ denotes the region of $\mathbb{V}_k(S)$ associated to the sites in H.

Lemma 1: If $p,\, q,\, r$ are points in $\mathbb{T}^2$, then:

$$d_{\mathbb{T}^2}(r,\, p) \leq_{\mathbb{T}^1} d_{\mathbb{T}^2}(r,\, q) \Longleftrightarrow d_{\mathbb{T}^2}(\neg r,\, p) \geq_{\mathbb{T}^1} d_{\mathbb{T}^2}(\neg r,\, q)$$

Proof: Let $[x_1, w_1]$ and $[x_2, w_2]$ be the coordinates, in $\mathbb{T}^1$, of $d_{\mathbb{T}^2}(r,\, p)$ and $d_{\mathbb{T}^2}(r,\, q)$, respectively. Then,

$d_{\mathbb{T}^2}(r,\, p) \leq_{\mathbb{T}^1} d_{\mathbb{T}^2}(r,\, q) \iff [x_1, w_1] \leq_{\mathbb{T}^1} [x_2, w_2]$

By the definition of $\leq_{\mathbb{T}^1}$,
$[x_1, w_1] \leq_{\mathbb{T}^1} [x_2, w_2] \iff$
either
$(i)\ w_1 w_2 \leq 0\ and\ w_1 \geq 0\ and\ w_2 \leq 0 \iff$
$(-w_1)(-w_2) \leq 0\ and\ w_1 \geq 0\ and\ w_2 \leq 0 \iff$
$(-w_2)(-w_1) \leq 0\ and\ (-w_2) \geq 0\ and\ (-w_1) \leq 0;$
or
$(ii)\ w_1 w_2 > 0\ and\ x_1 w_2 \leq x_2 w_1 \iff$
$(-w_1)(-w_2) > 0\ and\ x_1(-w_2) \geq x_2(-w_1) \iff$
$(-w_1)(-w_2) > 0\ and\ x_2(-w_1) \leq x_1(-w_2).$

Since $((i)$ or $(ii)) \iff [x_1, -w_1] \geq_{\mathbb{T}^1} [x_2, -w_2]$, it follows that

$$[x_1, w_1] \leq_{\mathbb{T}^1} [x_2, w_2] \iff [x_1, -w_1] \geq_{\mathbb{T}^1} [x_2, -w_2].$$

By the definition of two-sided Euclidean Plane distance:

$$d_{\mathbb{T}^2}(r,\, p) = [x_1, w_1] \iff d_{\mathbb{T}^2}(\neg r,\, p) = [x_1, -w_1].$$

Similarly,

$$d_{\mathbb{T}^2}(r,\, q) = [x_2, w_2] \iff d_{\mathbb{T}^2}(\neg r,\, q) = [x_2, -w_2].$$

Therefore, we have proved that
$d_{\mathbb{T}^2}(r,\, p) \leq_{\mathbb{T}^1} d_{\mathbb{T}^2}(r,\, q) \iff d_{\mathbb{T}^2}(\neg r,\, p) \geq_{\mathbb{T}^1} d_{\mathbb{T}^2}(\neg r,\, q).$

□

Theorem 1: The order k Voronoi diagram of a set S of n sites in $\mathbb{T}^2$ is antipodal to its order $n-k$ Voronoi diagram, $\forall k:\ 1 \leq k < n$.

Proof: Consider the order k Voronoi diagram of S, $\mathbb{V}_k(S)$, and a point $x \in \mathbb{T}^2$. Let H_k be a proper subset of S, containing k of its sites. By definition,
$x \in \mathcal{V}_k(H_k,\, S) \iff d_{\mathbb{T}^2}(x,\, p) \leq_{\mathbb{T}^1} d_{\mathbb{T}^2}(x,\, q),$
$\forall p \in H_k,\ \forall q \in S \setminus H_k$.
By Lemma 1,
$x \in \mathcal{V}_k(H_k,\, S) \iff d_{\mathbb{T}^2}(\neg x,\, p) \geq_{\mathbb{T}^1} d_{\mathbb{T}^2}(\neg x,\, q),$
$\forall p \in H_k,\ \forall q \in S \setminus H_k$,
which is equivalent to
$x \in \mathcal{V}_k(H_k,\, S) \iff d_{\mathbb{T}^2}(\neg x,\, q) \leq_{\mathbb{T}^1} d_{\mathbb{T}^2}(\neg x,\, p),$
$\forall q \in S \setminus H_k,\ \forall p \in H_k$,
or

$$x \in \mathcal{V}_k(H_k,\, S) \iff \neg x \in \mathcal{V}_{n-k}(S \setminus H_k,\, S).$$

□

The following lemma, whose Euclidian version can be found in [4], describes the proper points of $\mathbb{T}^2$ which lie on edges of some order k Voronoi diagram. For a description of points at infinity, see Lemma 3 and Theorem 4.

Lemma 2: Let $\mathbb{V}_k(S)$ be the order k Voronoi diagram of a set S of sites in $\mathbb{T}^2$. $p \notin \Omega$ is a point of an edge $\bar{B}_k(s_i,\, s_j)$ of $\mathbb{V}_k(S)$ if and only if the circle centered at p with radius $d_{\mathbb{T}^2}(p,\, s_i) = d_{\mathbb{T}^2}(q,\, s_j)$ contains $k-1$ sites of S in its interior.

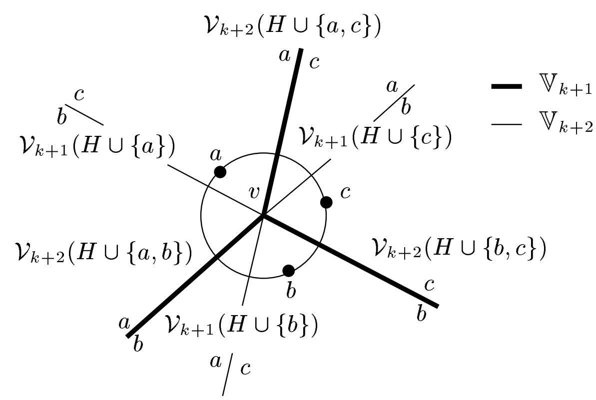
Proof: It is a simple generalization of the proof of lemma 3 due to Lee [4].

□

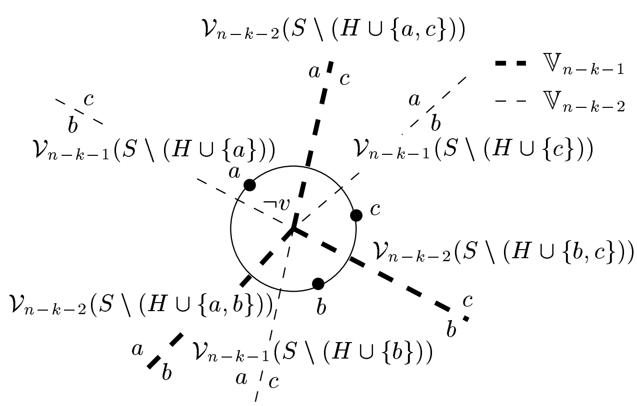
Theorem 2: Let $v,\, \neg v \in \mathbb{T}^2$ be the proper circumcenters of $s_a,\, s_b,\, s_c \in S$, with $s_a,\, s_b,\, s_c$ on the same range of $\mathbb{T}^2$. Let H denote the set of sites that are closer to v than $s_a,\, s_b,\, s_c$ are:

$$H = \{z \in S :\ d_{\mathbb{T}^2}(v,\, z) <_{\mathbb{T}^1} d_{\mathbb{T}^2}(v,\, s_a)\}$$

and let $k = |H|$. Then, v is a Voronoi vertex of $\mathbb{V}_{k+1}(S)$ and $\mathbb{V}_{k+2}(S)$, while $\neg v$ is a Voronoi vertex of $\mathbb{V}_{n-k-1}(S)$ and $\mathbb{V}_{n-k-2}(S)$. Furthermore, the edges and regions that are incident on v and $\neg v$ are given on the diagrams shown in figures 1 and 2, respectively.



(a)



(b)

Figure 1: Incident edges: (a) v on the front range and (b) $\neg v$ on the back range.

Proof: It is a generalization of the proof of theorem 1 due to

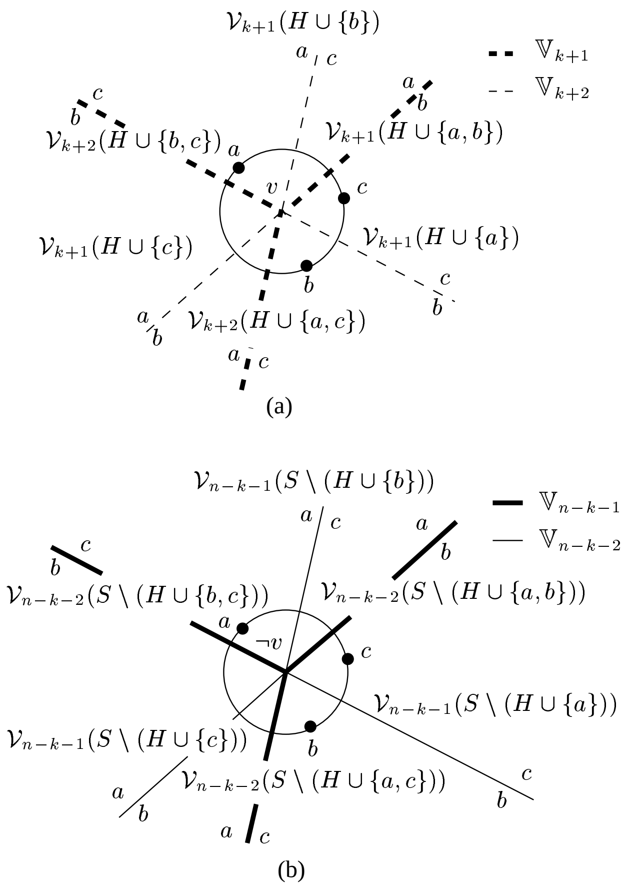


Figure 2: Incident edges: (a) v on the back range and (b) $\neg v$ on the front range.

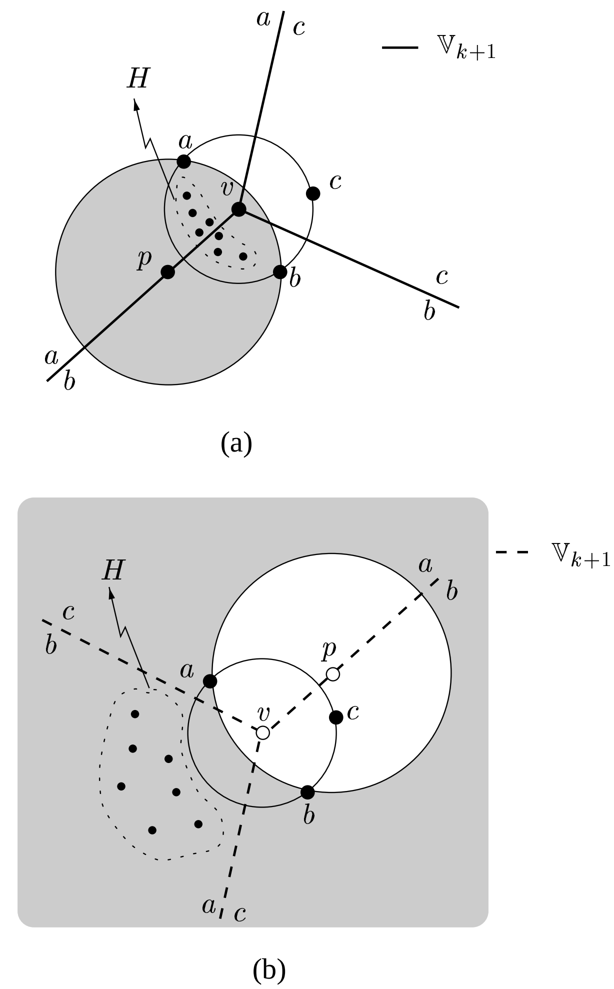


Figure 3: Edges of $\mathbb{V}_{k+1}$ incident on v: (a) v on the front range and (b) v on the back range. The shaded area represents the interior of the circle centered at p. Note that, in both cases, this circle contains $|H| = k$ sites in its interior.

Dehne [1]. By Theorem 1, it follows that the edges incident on $\neg v$ are the edges antipodal to those incident on v.

□

Theorem 3: Let $s_a, s_b, s_c \in S$ and $v \in \mathbb{T}^2 \setminus \Omega$ be a Voronoi vertex of $\mathbb{V}_i(S)$ so that:

$$v \in \mathcal{V}_i(A, S) \cap \mathcal{V}_i(B, S) \cap \mathcal{V}_i(C, S),$$

$\neg v$ is a Voronoi vertex of $\mathbb{V}_j(S)$,

$$\neg v \in \mathcal{V}_j(A', S) \cap \mathcal{V}_j(B', S) \cap \mathcal{V}_j(C', S),$$

and v and $\neg v$ are the proper circumcenters of s_a, s_b, s_c. Let H denote the set of sites of S that are closer to v than s_a, s_b, s_c are:

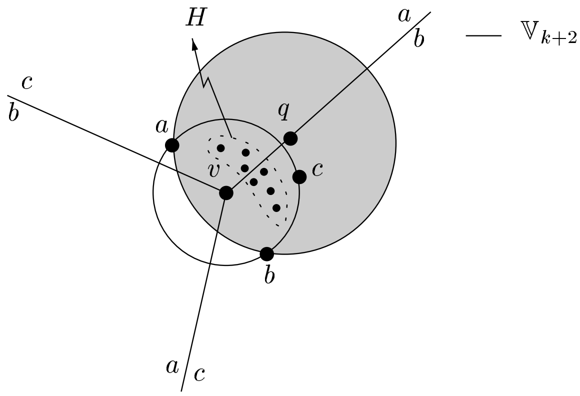
$$H = \{z \in S : d_{\mathbb{T}^2}(v, z) <_{\mathbb{T}^1} d_{\mathbb{T}^2}(v, s_a)\}$$

and let $k = |H|$. Then, either
$i = k + 1$ and $j = n - k - 1$ with
$\{A, B, C\} = \{H \cup \{s_a\}, H \cup \{s_b\}, H \cup \{s_c\}\}$ and
$\{A', B', C'\} = \{S \setminus (H \cup \{s_a\}), S \setminus (H \cup \{s_b\}), S \setminus (H \cup \{s_c\})\}$
or
$i = k + 2$ and $j = n - k - 2$ with
$\{A, B, C\} = \{H \cup \{s_a, s_b\}, H \cup \{s_b, s_c\}, H \cup \{s_a, s_c\}\}$
and
$\{A', B', C'\} = \{S \setminus (H \cup \{s_a, s_b\}), S \setminus (H \cup \{s_b, s_c\}), S \setminus (H \cup \{s_a, s_c\})\}$.

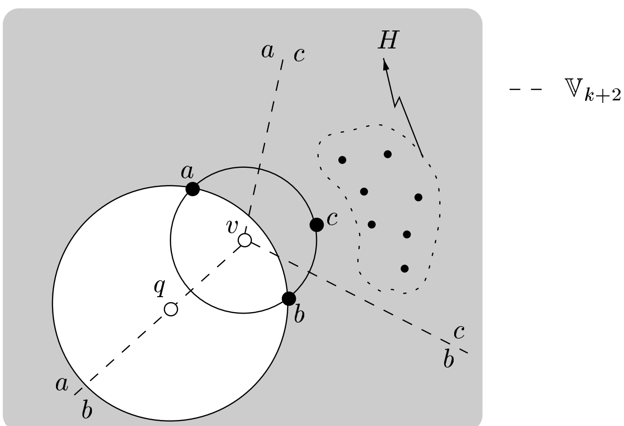
Proof: It follows as a generalization of theorem 2 in [1]. By Theorem 1, we have that $A' = S \setminus A$, $B' = S \setminus B$ and $C' = S \setminus C$.

□

Theorem 2 presents a sufficient condition for a point



(a)



(b)

Figure 4: Edges of $\mathbb{V}_{k+2}$ incident on v: (a) v on the front range and (b) v on the back range. The circle centered at q, in both cases, contains $|H|+1 = k+1$ sites in its interior.

$v \notin \Omega$ to be a Voronoi vertex of $\mathbb{V}_{k+1}(S)$ and $\mathbb{V}_{k+2}(S)$. On the other hand, Theorem 3 shows that the condition is necessary. Hence, Theorem 2 describes all proper vertices and its incident edges and regions of all diagrams $\mathbb{V}_k(S)$, $\forall k : 1 \le k \le n-1$. Before we describe vertices at infinity, we need the following definition.

Definition 3: Consider the identity homeomorphism h which maps the spherical model of $\mathbb{T}^2$ to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Let $B(q,\, \alpha)$ denote the set of points of $\mathbb{S}^2$ whose $\mathbb{R}^3$ distance from $q \in \mathbb{S}^2$ is less than or equal to α. The *neighborhood* $N(p, \alpha)$ on $\mathbb{T}^2$ is the set of points $h^{-1}(r)$, with $r \in B(h(p),\, \alpha)$.

The following lemma characterizes the points which lie on edges of some order k Voronoi diagram. Corollary 1, which follows from Lemma 3, is used in Theorem 5 for characterizing the vertices at infinity.

Lemma 3: If $p \in \mathbb{T}^2$ is a point which borders two adjacent Voronoi regions $\mathcal{V}_k(H \cup \{s_a\})$ and $\mathcal{V}_k(H \cup \{s_b\})$ of $\mathbb{V}_k(S)$, then p is contained in the line

$$r = norm(dir(s_a \vee s_b)) \vee midpoint(s_a,\, s_b).$$

Proof: Consider a point $p \in \mathbb{T}^2$ which borders two adjacent Voronoi regions $\mathcal{V}_k(H \cup \{s_a\})$ and $\mathcal{V}_k(H \cup \{s_b\})$ of $\mathbb{V}_k(S)$. If $p \notin \Omega$, then $d_{\mathbb{T}^2}(p,\, s_a) = d_{\mathbb{T}^2}(p,\, s_b)$ and the kth nearest neighbor of p is s_a or s_b. Since $p \notin \Omega$, p is contained in r. If $p \in \Omega$, by contradiction, suppose that p is not in r. Then, either
(i) s_a and s_b are on different ranges, in which case $r = \Omega$, which contains p, a contradiction to $p \notin r$;
or
(ii) s_a and s_b are on the same range, in which case $p \in \Omega$ and $p \notin r$. If we take $\alpha > 0$ sufficiently small, the neighborhood $N(p, \alpha)$ contains points of only one region, a contradiction to the fact that p borders $\mathcal{V}_k(H \cup \{s_a\})$ and $\mathcal{V}_k(H \cup \{s_b\})$. Again, we have a contradiction.

□

Corollary 1: If $v \in \Omega$ is a vertex of $\mathbb{V}_k(S)$ then there exist $s_a,\, s_b \in S$ with s_a and s_b on the same range of $\mathbb{T}^2$, such that

$$v \in r = norm(dir(s_a \vee s_b)) \vee midpoint(s_a,\, s_b).$$

Let $s_a,\ s_b,\ s_c \in S$. We denote by $M(a,b,c)$ and $\neg M(a,b,c)$ the circumcenters of $s_a,\, s_b,\, s_c$. Since all sites are cocircular if we take any point on Ω as their circumcenter, there is no site in the interior of a circle centered at a point on Ω. Theorem 2 is not enough to characterize to which Voronoi diagrams a vertex on Ω belongs. Theorems 4 and 5 describe which vertex $M(a,b,j)$, circumcenter of $s_a,\, s_b,\, s_j \in S$, is adjacent to a vertex $v \in \Omega$ and to which Voronoi diagrams v belongs. Firstly, let us establish sufficient conditions.

Theorem 4: Let $v \in \Omega$ and $s_a,\, s_b \in S$, with s_a and s_b on the same range of $\mathbb{T}^2$, such that

$$v \in r = norm(dir(s_a \vee s_b)) \vee midpoint(s_a,\, s_b).$$

If there exists at least one site on the opposite range of s_a and s_b, then it is sufficient for v to be a vertex of $\mathbb{V}_k(S)$, $1 \le k < n$, that either
$(i)\, \exists\, s_j \in S \setminus \{s_a,\, s_b\} :\ M(a,b,j)$ is a proper vertex of $\mathbb{V}_k(S)$, such that there are no other vertices of $\mathbb{V}_k(S)$ in the segment from $M(a,b,j)$ to v,
or

$(ii)\, \exists\, s_j \in S \setminus \{s_a,\, s_b\}\, :\, M(a,b,j)$ is a vertex of $\mathbb{V}_k(S)$ and $\forall\, s_j \in S \setminus \{s_a,\, s_b\}$, with $M(a,b,j) \in \mathbb{V}_k(S)$, we have $M(a,b,j) \in \Omega$.

Proof: Let $s_a,\, s_b \in S$, with s_a and s_b on the same range of $\mathbb{T}^2$, such that

$$v \in r = norm(dir(s_a \vee s_b)) \vee midpoint(s_a,\, s_b).$$

Consider that there exists at least one site on the range opposite to s_a and s_b.
Assume that case (i) occurs. Since $M(a,b,j)$ is a vertex of $\mathbb{V}_k(S)$, then it borders at least three regions, among them $\mathcal{V}_k(H \cup \{s_a\})$ and $\mathcal{V}_k(H \cup \{s_b\})$. Since there are no other vertices of $\mathbb{V}_k(S)$ in the segment from $M(a,b,j)$ to v, these regions are also adjacent to v. Thus, v borders $\mathcal{V}_k(H \cup \{s_a\})$ and $\mathcal{V}_k(H \cup \{s_b\})$. Now, consider a neighborhood $N(v,\alpha)$. If α is sufficiently small, then the portion of $N(v,\alpha)$ that is on the range opposite to $M(a,b,j)$ contains points which are closer to a point s_c on that range than s_a and s_b are. Therefore, s_c is among the k nearest neighbors of these points, i.e, there exists a region $\mathcal{V}_k(H' \cup \{s_c\})$, with $s_a \notin H'$ and $s_b \notin H'$, such that v also borders $\mathcal{V}_k(H' \cup \{s_c\})$. Therefore, v is a vertex of $\mathbb{V}_k(S)$.

Assume now that case (ii) occurs. Since $v \in \Omega$ and $v \in r$, v is either $M(a,b,j)$ or $\neg M(a,b,j)$. If v is $M(a,b,j)$ then it is a vertex. Otherwise, since $M(a,b,j)$ is a vertex of $\mathbb{V}_k(S)$ and $\forall\, s_j \in S \setminus \{s_a,\, s_b\}$, with $M(a,b,j) \in \mathbb{V}_k(S)$, we have $M(a,b,j) \in \Omega$, $M(a,b,j)$ has an adjacent vertex on $r \wedge \Omega$, which is not $M(a,b,j)$. Thus, it is $\neg M(a,b,j)$, i.e, $\neg M(a,b,j)$ is a vertex of $\mathbb{V}_k(S)$. Since v is either $M(a,b,j)$ or $\neg M(a,b,j)$, but v is not $M(a,b,j)$, then v is $\neg M(a,b,j)$, which is a vertex of $\mathbb{V}_k(S)$.

$\square$

Note that we assumed that $S \cup \neg S$ contains no four points with proper circumcenter. Therefore, it is easy to see that, in case (ii) of Theorem 4, the only vertices on r are v and $\neg v$. We now show the converse of Theorem 4.

Theorem 5: Let $v \in \Omega$ be a vertex of $\mathbb{V}_k(S)$. Then there exist $s_a,\, s_b \in S$ with s_a and s_b on the same range of $\mathbb{T}^2$ such that

$$v \in r = norm(dir(s_a \vee s_b)) \vee midpoint(s_a,\, s_b)$$

and there exists at least one site on the range opposite to s_a and s_b. We also have either
$(i)\, \exists\, s_j \in S \setminus \{s_a,\, s_b\}\, :\, M(a,b,j)$ is a proper vertex of $\mathbb{V}_k(S)$, such that there are no other vertices of $\mathbb{V}_k(S)$ in the segment from $M(a,b,j)$ to v,
or
$(ii)\, \exists\, s_j \in S \setminus \{s_a,\, s_b\}\, :\, M(a,b,j)$ is a vertex of $\mathbb{V}_k(S)$ and $\forall\, s_j \in S \setminus \{s_a,\, s_b\}$, with $M(a,b,j) \in \mathbb{V}_k(S)$, we have $M(a,b,j) \in \Omega$.

Proof: By Corollary 1, if $v \in \Omega$ is a vertex of $\mathbb{V}_k(S)$ then $\exists\, s_a,\, s_b \in S$ with s_a and s_b on the same range of $\mathbb{T}^2$ such that

$$v \in r = norm(dir(s_a \vee s_b)) \vee midpoint(s_a,\, s_b)$$

- If s_a and s_b are the only sites on its range, then either:

1. There is no site on the opposite range, $S = \{s_a, s_b\}$. Thus, the only Voronoi diagram is $V_1(\{s_a, s_b\})$ which is just line r, i.e, there are no vertices, a contradiction to v being a vertex of some $V_k(S)$; or
2. There exists at least one site on the opposite range of s_a and s_b. Thus, $M(a,b,j) \in \Omega,\ \forall\, s_j \in S \setminus \{s_a, s_b\}$. Since v is a vertex of $\mathbb{V}_k(S)$ and v is a circumcenter of $s_a,\, s_b$ and any site on its opposite range, $\exists\, s_j \in S \setminus \{s_a,\, s_b\}\, :$ $M(a,b,j)$ is a vertex of $\mathbb{V}_k(S)$. This implies (ii).

- If there exists at least one site on the same range as s_a and s_b, then either:

1. There is no site on the opposite range, in which case $S = S' \cup \{s_a, s_b\}$, with all sites of S' on the same range as s_a and s_b. Therefore, there is no vertex on Ω, a contradiction to v being a vertex of some $V_k(S)$; or
2. There exists at least one site s_c on the range opposite to s_a and s_b. So v is an intersection of r and Ω, where Ω can be regarded as the bisector of s_c and any other site on the same range of s_a and s_b. By contradiction, assume that (i) is false. Let $S'' \subset S$ be the set of sites such that $M(a,b,s')$, with $s' \in S''$, is a vertex of $\mathbb{V}_k(S)$ in the segment from $M(a,b,j)$ to v. Consider the vertex $M(a,b,s'_1) \in \mathbb{V}_k(S)$, with $s'_1 \in S''$. Since (i) was assumed to be false, the segment from $M(a,b,s'_1)$ to v contains another circumcenter $M(a,b,s'_2) \in \mathbb{V}_k(S)$, with $s'_2 \in S''$. In the same way, the segment from $M(a,b,s'_2)$ to v must contain another circumcenter $M(a,b,s'_3) \in \mathbb{V}_k(S)$, with $s'_3 \in S''$. This implies that the number of circumcenters is infinite and therefore S would also be an infinite set, a contradiction. This implies (i).

$\square$

Below, we show that the order k Voronoi diagram of a finite set of sites has an exact number of regions. Before that, we present some definitions and properties used in the proof.

Definition 4: In the order $k+1$ Voronoi diagram, we refer to v as a *close-type vertex*, while in the order $k+2$ diagram v is a *far-type vertex*. See Theorem 2.

Definition 5: An edge e is a *close-type edge* of $\mathbb{V}_i(S)$ if and only if both vertices of e are close-type in $\mathbb{V}_i(S)$.

Definition 6: An edge e is a *far-type edge* of $\mathbb{V}_i(S)$ if and only if e is incident on at least one far-type vertex of $\mathbb{V}_i(S)$.

Definition 7: $\mathcal{V}_{i+1}(H)$ is a *type I region* of $\mathbb{V}_{i+1}(S)$ if and only if $\mathcal{V}_{i+1}(H)$ contains in its interior only one edge e of $\mathbb{V}_i(S)$, and e is close-type in $\mathbb{V}_i(S)$.

Definition 8: $\mathcal{V}_{i+1}(H)$ is a *type II region* of $\mathbb{V}_{i+1}(S)$ if and only if $\mathcal{V}_{i+1}(H)$ contains in its interior only far-type vertices and far-type edges of $\mathbb{V}_i(S)$.

Lemma 4 guarantees that no other types of regions can occur. Lemmas 4 and 5 are generalizations of lemmas 8 and 10 in [4], respectively.

Lemma 4: Every region of any Voronoi diagram $\mathbb{V}_i(S)$ is either a type I region or a type II region.

Lemma 5: If $\mathcal{V}_i(H)$ contains m far-type vertices of $\mathbb{V}_{i-1}(S)$ in its interior, then it also contains $2m+1$ far-type edges of $\mathbb{V}_{i-1}(S)$.

The following lemma establishes a relation between the number of vertices, edges and regions of any order k Voronoi diagram. Lemma 7 shows the exact number of regions of the order k Voronoi diagram of a finite set of sites. This is an intrinsic property of $\mathbb{T}^2$.

Lemma 6: Let V_k denote the number of vertices, E_k the number of edges and F_k the number of regions of $\mathbb{V}_k(S)$, $1 \leq k \leq n-1$. Then $V_k = 2F_k - 4$ and $E_k = 3F_k - 6$.

Proof: Since F_k is the number of faces of the subdivision corresponding to $\mathbb{V}_k(S)$ and each vertex of $\mathbb{V}_k(S)$ has degree 3. Then, by Euler's formula, $E_k + 2 = V_k + F_k$. Since $E_k = \frac{3}{2}V_k$, we have $V_k = 2F_k - 4$ and $E_k = 3F_k - 6$. $\square$

Lemma 7: The number F_k of regions of $\mathbb{V}_k(S)$, $1 \leq k \leq n-1$, is

$$F_k = 2k(n-k) - n + 2$$

Proof: Let E_i be the number of edges and V_i the number of vertices of $\mathbb{V}_i(S)$. Let V'_{i+1} denote the number of close-type vertices of $\mathbb{V}_{i+1}(S)$, i.e., the number of vertices of $\mathbb{V}_{i+1}(S)$ that do not exist in $\mathbb{V}_i(S)$. Then, $V_{i+1} = V'_{i+1} + V'_i$ or

$$V'_{i+1} = V_{i+1} - V'_i. \qquad (1)$$

Let E'_{i+1} denote the number of close-type edges (edges connecting only close-type vertices) and E''_{i+1} the number of far-type edges (edges incident on at least one far-type vertex of $\mathbb{V}_{i+1}(S)$). We have $E_1 = E'_1$ and

$$E_{i+1} = E'_{i+1} + E''_{i+1}.$$

Since each close-type edge of $\mathbb{V}_{i+1}(S)$ corresponds to a type I region of $\mathbb{V}_{i+2}(S)$, the number F'_{i+2} of type I regions of $\mathbb{V}_{i+2}(S)$ is equal to the number E'_{i+1} of close-type edges of $\mathbb{V}_{i+1}(S)$. Let F''_{i+2} denote the number of type II regions of $\mathbb{V}_{i+2}(S)$ and let m_j be the number of far-type vertices of $\mathbb{V}_{i+1}(S)$ contained in the interior of the jth type II region of $\mathbb{V}_{i+2}(S)$. Let V''_{i+1} denote the total number of far-type vertices of $\mathbb{V}_{i+1}(S)$. Then, we have:

$$\sum_{j=1}^{F''_{i+2}} m_j = V''_{i+1} = V'_i$$

By Lemma 5, the number e_j of edges incident on these m_j vertices is $e_j = 2m_j + 1$. Then,

$$E''_{i+1} = \sum_{j=1}^{F''_{i+2}} e_j = 2\sum_{j=1}^{F''_{i+2}} m_j + F''_{i+2} = 2V'_i + F''_{i+2}$$

or $F''_{i+2} = E''_{i+1} - 2V'_i$. Therefore,

$$\begin{aligned} F_{i+2} &= F'_{i+2} + F''_{i+2} \\ &= E'_{i+1} + E''_{i+1} - 2V'_i \\ &= E_{i+1} - 2V'_i \end{aligned}$$

Since $F_1 = n$, we have $F_2 = E_1 - 2V'_0 = E_1 = 3F_1 - 6 = 3n - 6$, by Lemma 6. Furthermore, $F_{k+2} = E_{k+1} - 2V'_k = 3F_{k+1} - 6 - 2V'_k$.
By (1), $V'_k = V_k - V'_{k-1}$, and $V'_1 = V_1 = 2F_1 - 4 = 2n - 4$, by Lemma 6. Then,

$$V'_k = V_k - (V_{k-1} - V'_{k-2}) = (V_k - V_{k-1}) + V'_{k-2}.$$

Therefore, $V'_k = \sum_{i=\lceil k/2 \rceil + 1}^{k} (V_{2i-k} - V_{2i-k-1}) + V_{2\lceil k/2 \rceil - k}$ $= \sum_{i=1}^{k} [(-1)^{k-i} V_i]$, i.e.,

$$V'_k = \sum_{i=1}^{k} [(-1)^{k-i}(2F_i - 4)].$$

Then,

$$F_{k+2} = 3F_{k+1} - 6 - 2\sum_{i=1}^{k} [(-1)^{k-i}(2F_i - 4)]. \qquad (2)$$

By induction on k, we can prove that:

$$F_k = 2k(n-k) - n + 2$$

Basis: $F_1 = n$ and $F_2 = 3n - 6$.
Hypothesis (1): $\exists\, j \geq 2$ such that

$$F_k = 2k(n-k) - n + 2,\ \forall k : 1 \leq k \leq j$$

Induction Step: By (2),

$$F_{k+1} = 3F_k - 6 - 2\sum_{i=1}^{k-1}[(-1)^{k-1-i}(2F_i - 4)].$$

By induction on k, we can prove that:

$$\sum_{i=1}^{k-1}[(-1)^{k-1-i}(2F_i - 4)] = F_k - n + 2k - 2$$

Basis: $k = 2$:

$$\sum_{i=1}^{1}[(-1)^{1-i}(2F_i - 4)] = 2F_1 - 4 =$$

$$= 2F_1 - 4 = 2n - 4 = F_2 - n + 2(2) - 2$$

Hypothesis (2): $\exists\, k \geq 2$:

$$\sum_{i=1}^{k-1}[(-1)^{k-1-i}(2F_i - 4)] = F_k - n + 2k - 2$$

Induction Step:

$$\sum_{i=1}^{k}[(-1)^{k-i}(2F_i - 4)] =$$

$$= (-1)^{k-k}(2F_k - 4) + \textstyle\sum_{i=1}^{k-1}[(-1)^{k-i}(2F_i - 4)]$$

$$= 2F_k - 4 - \sum_{i=1}^{k-1}[(-1)^{k-1-i}(2F_i - 4)]$$

By hypothesis (2),

$$\sum_{i=1}^{k}[(-1)^{k-i}(2F_i - 4)] =$$

$= 2F_k - 4 - (F_k - n + 2k - 2)$
$= F_k + n - 2k - 2$
$= 2k(n-k) - n + 2 + n - 2k - 2$, by hypothesis (1)
$= 2k(n-k) - 2k$
$= 2(k+1)(n-k) - 2(n-k) - 2k$
$= 2(k+1)(n-k-1) + 2(k+1) - 2n$
$= [2(k+1)(n-(k+1)) - n + 2] - n + 2(k+1) - 2$
$= F_{k+1} - n + 2(k+1) - 2$

□

Then,

$$F_{k+1} = 3F_k - 6 - 2[F_k - n + 2k - 2] = F_k + 2n - 4k - 2$$

By hypothesis (1),
$F_{k+1} = 2k(n-k) - n + 2 + 2n - 4k - 2$
$= 2k(n-k) + n - 4k$
$= 2k(n-k-1) - 2k + n$
$= 2(k+1)(n-k-1) - 2(n-k-1) - 2k + n$
$= 2(k+1)(n-(k+1)) - n + 2$

□

Theorem 6: Let V_k denote the number of vertices, E_k the number of edges and F_k the number of regions of $\mathbb{V}_k(S)$, $1 \leq k \leq n-1$, then we have:
(i) $\Theta(V_k) = \Theta(E_k) = \Theta(F_k), \forall k : 1 \leq k \leq n-1$
(ii) $F_k \in O(k(n-k)) \subset O(kn)$, $\forall k : 1 \leq k \leq n-1$
(iii) $\sum_{k=1}^{n-1} F_k \in \Theta(n^3)$
(iv) $\sum_{k=1}^{n-1} kF_k \in \Theta(n^4)$

Proof: It is a generalization of theorem 3 in [1].

□

4 The Algorithm

In this section, we present an algortihm for constructing all order k Voronoi diagrams $\mathbb{V}_k(S)$. An array L of all circumcenters is constructed and with each circumcenter we maintain their incident edges and regions. Each $\mathbb{V}_k(S)$ is represented by a linked list of vertices, each vertex corresponding to a circumcenter in L.

1. Construct an array L of all circumcenters $M(a, b, c)$ and $\neg M(a, b, c)$, $a > b > c$, each of which defined by three sites s_a, s_b and s_c of S. Store $M(a, b, c)$, $\neg M(a, b, c)$ and (a, b, c) on L at address $\binom{a-1}{3} + \binom{b-1}{2} + c$.

2. Traverse L and calculate, for each $M(a, b, c)$, the set
$$H(a, b, c) = \{z \in S : d_{\mathbb{T}^2}(M(a,b,c), z) <_{\mathbb{T}^1} d_{\mathbb{T}^2}(M(a,b,c), s_a)\}$$
and the rays and regions incident on $M(a, b, c)$, as described in Theorem 2. For $\neg M(a, b, c)$, Theorem 1 implies that the set $\neg H(a, b, c) = S \setminus H(a, b, c) \setminus \{s_a, s_b, s_c\}$. If $|H(a, b, c)| = k$ then $H(a, b, c)$ is the set of k sites that are closer to $M(a, b, c)$ than s_a, s_b and s_c are, while $\neg H(a, b, c)$ is the set of $n - k - 3$ sites that are closer to $\neg M(a, b, c)$ than s_a, s_b and s_c are. Add $M(a, b, c)$ to the end of the corresponding linked lists of $\mathbb{V}_{k+1}(S)$ and $\mathbb{V}_{k+2}(S)$. Similarly, add $\neg M(a, b, c)$ to the end of the corresponding linked lists of $\mathbb{V}_{n-k-2}(S)$ and $\mathbb{V}_{n-k-1}(S)$. Store with $M(a, b, c)$ in L, the address of $M(a, b, c)$ and $\neg M(a, b, c)$ in such lists.

3. Traverse L again to link the vertices of the same $\mathbb{V}_k(S)$. Each $M(a, b, c)$ is a vertex in the lists of $\mathbb{V}_i(S)$ and $\mathbb{V}_{i+1}(S)$, with 6 incident rays. For each ray r, which

is a portion of $B(s_a,\, s_b)$, incident on $M(a,\, b,\, c)$ of $\mathbb{V}_j(S)$, check whether $M(a,\, b,\, d)$, with $s_d \in S \setminus \{s_a,\, s_b,\, s_c\}$, is a vertex of $\mathbb{V}_j(S)$, such that $M(a,\, b,\, d) \subset B(s_a,\, s_b)$. If there are more such $M(a,\, b,\, d)$, take the one with the smallest distance from $M(a,\, b,\, c)$. If $M(a,\, b,\, c) \in \Omega$ and $M(a,\, b,\, d) \in \Omega$, then we take the one in the smallest neighborhood $N(M(a,b,c),\alpha)$. If $M(a,\, b,\, c) \notin \Omega$ and $M(a,\, b,\, d) \in \Omega$ then we check the cases described in Theorem 5. Reduce r to an edge $(M(a,\, b,\, c),\, M(a,\, b,\, d))$ and the corresponding ray of $M(a,\, b,\, d)$ to an edge $(M(a,\, b,\, d),\, M(a,\, b,\, c))$.

4.1 Correctness

Step 1 computes all circumcenters $M(a,\, b,\, c)$ defined by three sites $s_a,\, s_b$ and s_c of S. Step 2 calculates the set H of the k sites that are closer to $M(a,\, b,\, c)$ than $s_a,\, s_b$ and s_c are. By Theorem 2, each $M(a,\, b,\, c) \notin \Omega$ corresponds to a vertex of $\mathbb{V}_{k+1}(S)$ and $\mathbb{V}_{k+2}(S)$. By Theorem 3, these are all the proper vertices of all diagrams $\mathbb{V}_k(S)$. Step 3 just links the adjacent vertices. If $M(a,\, b,\, c)$ and $M(a,\, b,\, d) \notin \Omega$ then the correctness follows from Theorem 3. It still remains to prove that the vertices at infinity are correctly linked. By Theorem 5, if a vertex $M(a,\, b,\, c) \in \Omega$, then either:
(i) it has an adjacent vertex $M(a,b,j)$ not in Ω or
(ii) all of its adjacents vertices $M(a,b,j) \in \Omega$.

In case (i), $M(a,b,c)$ is a vertex adjacent to $M(a,b,j)$. Since $M(a,b,j) \notin \Omega$ they are linked in step 3. In case (ii), the vertex adjacent to $M(a,b,c)$ in $\mathbb{V}_k(S)$ is the one with the smallest neighborhood $N(M(a,b,c),\alpha)$. It is linked to $M(a,\, b,\, c)$ in step 3.

By Theorem 4, these are all the improper vertices of all diagrams $\mathbb{V}_k(S)$. Since all the proper or improper vertices of all diagrams $\mathbb{V}_k(S)$ are computed and correctly linked, we have proved the correctness of the algorithm.

4.2 Space and Time Complexity

The space complexity of the algorithm is

$$\Theta(\sum_{k=1}^{n-1} kV_k) = \Theta(\sum_{k=1}^{n-1} kF_k) = \Theta(n^4).$$

Each circumcenter is computed in constant time. Since there are $\Theta(n^3)$ circumcenters, step 1 takes $\Theta(n^3)$ time. In the second step, for each of these circumcenters, the algorithm does $\Theta(n)$ iterations, i.e., step 2 takes $\Theta(n^4)$ time. In the same way, step 3 takes $\Theta(n^4)$ time. Therefore, the algorithm takes $\Theta(n^4)$ time and uses $\Theta(n^4)$ space and, by Theorem 6, it is optimal.

5 Concluding Remarks

We have presented a generalization for $\mathbb{T}^2$ of an algorithm to construct all order k Voronoi diagrams in $\mathbb{R}^2$. We have also shown that, for fixed k and for a finite set of sites, an order k Voronoi diagram in $\mathbb{T}^2$ has an exact number of regions.

We note that the correctness proof of this generalized algorithm for $\mathbb{T}^2$ is more complex than for the Euclidean plane. However, this generalization maintains the same simplicity of the original algorithm.

We have implemented the algorithm and it is useful in visualizing all the diagrams, since we do not have to construct each one independently. Figure 5 shows the order 2 Voronoi diagram of a set of points on the front range. Note that, as opposed to what happens in $\mathbb{R}^2$, even on unbounded regions of the diagram in $\mathbb{T}^2$ every edge is adjacent to two vertices, i.e. unbounded regions do not have to be considered special cases. Figure 6 shows the order 2 diagram for a set containing points on the front and on the back range. Note the edges and vertices on the line at infinity.

Even though the algorithm described in this paper constructs all order k Voronoi diagrams in $\mathbb{T}^2$ and has remarkable simplicity, it does not allow for the construction of a *single* diagram. As further studies in this topic, we are now working on the design of algorithms for constructing each order k Voronoi diagram in $\mathbb{T}^2$ independently.

Acknowledgments

This research was partially funded by Conselho Nacional de Desenvolvimento Científico e Tecnológico — CNPq (grants 132357/98-4 and 300157/90-8); by Fundação de Amparo à Pesquisa do Estado de São Paulo — FAPESP (grant 98/12955-3); and by Financiadora de Estudos e Projetos — Finep/MCT, Pronex: “Sistemas Avançados de Informação” (grant 76.97.1022.00).

References

[1] DEHNE, F. *An $O(n^4)$ Algorithm to Construct All Voronoi Diagrams for k Nearest Neighbor Searching*. Springer-Verlag, Lecture Notes in Computer Science, **154**, 1983.

[2] GON, C. *Computação Exata em Geometria Projetiva Orientada e Tratamento de Degenerações*. Master’s Thesis, Instituto de Computação – UNICAMP, June 1996.

[3] GON, C. and REZENDE, P. J. de *Um ambiente distribuído de visualização com suporte para geometria*

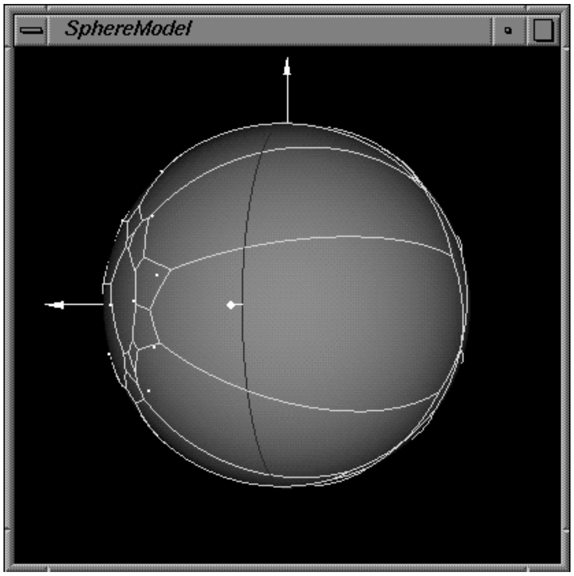


Figure 5: The order 2 Voronoi diagram of a set of points on the front range. Image generated with GeoPrO [3]

projetiva orientada. Proc. of the *IX SIBGRAPI*, pp. 71–78. 1996.

[4] LEE, D. T. *On k-Nearest Neighbor Voronoi Diagrams in the Plane*. IEEE Transactions on Computers, **c–31**, n. 6, June 1982.

[5] PINTO, G. A. and REZENDE, P. J. de, *Representation of conics in the oriented projective plane*. Proc. of the *X SIBGRAPI*, pp. 71–78. IEEE, 1997.

[6] PINTO, G. A. *Generalizações do Diagrama de Voronoi construídas através de Cônicas no Plano Projetivo Orientado e suas Visualizações*. Master's Thesis, Instituto de Computação – UNICAMP, May 1998.

[7] REZENDE, P. J. de and STOLFI, J. *Fundamentos de Geometria Computacional*. IX Escola de Computação, 1994.

[8] STOLFI, J. *Oriented Projective Geometry: a framework for geometric computations*. Academic Press Inc., 1991.

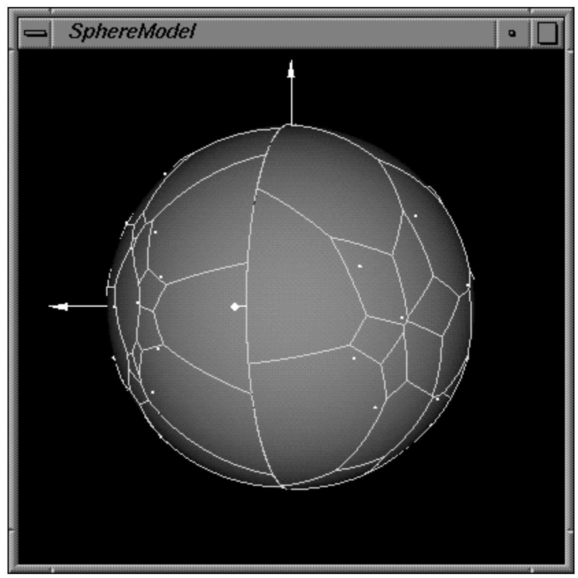


Figure 6: The order 2 Voronoi diagram for a set containing points on the front and on the back range. Image generated with GeoPrO [3]