CHARACTERIZATION OF LINEAR AND MORPHOLOGICAL OPERATORS

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CONTENT

INTRODUCTION

(1/1)

Linear and morphological operators are important pieces in image and signal processing.

Linear operators are models for many optical sensors and they can be used as filters for **sensor simulation** and **image restoration**.

Combinations of morphological operators can be used as filters for **image segmentation**.

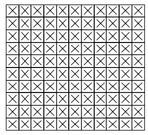
Operator characterization shows where are the differences and the similarities between the linear and morphological operators.

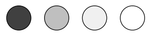
Operator characterization helps to understand how these operators are built.

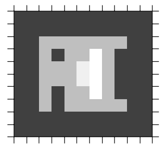
IMAGE PROCESSING OBJECTS

(1/4)

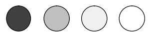
$f: \; E \rightarrow K$ (image)

$E \triangleq$ 

$K \triangleq \{$  $\}$

$f: E \rightarrow K \;\triangleq$ 

$l: \; K_1 \rightarrow K_2$ (gray–scale transform or lut)

$K \triangleq \{$  $\}$

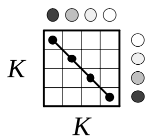
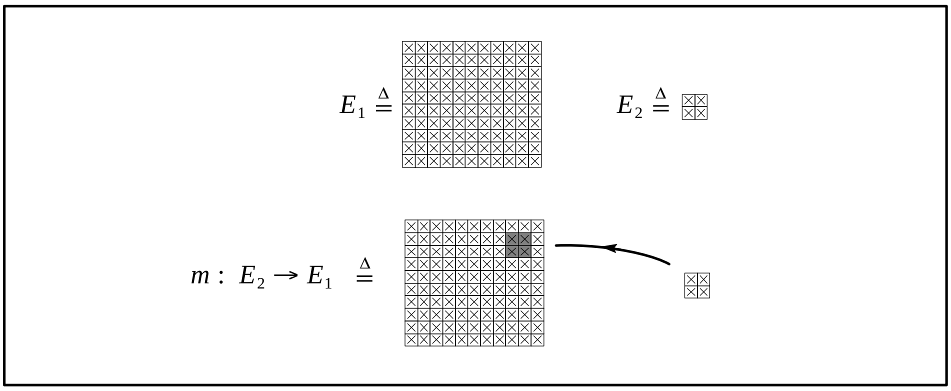
$l: K \rightarrow K \;\triangleq$ 

IMAGE PROCESSING OBJECTS

(2/4)

$$m: \; E_2 \to E_1$$ (domain transform)



$$\mu: \; K_1^E \to K_2$$ (measure)

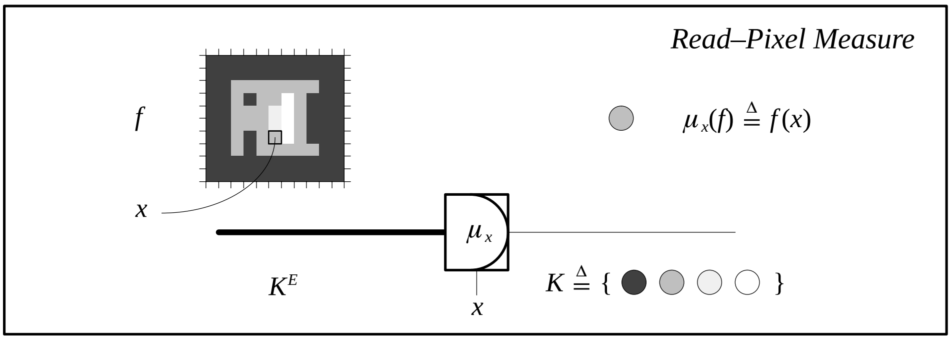
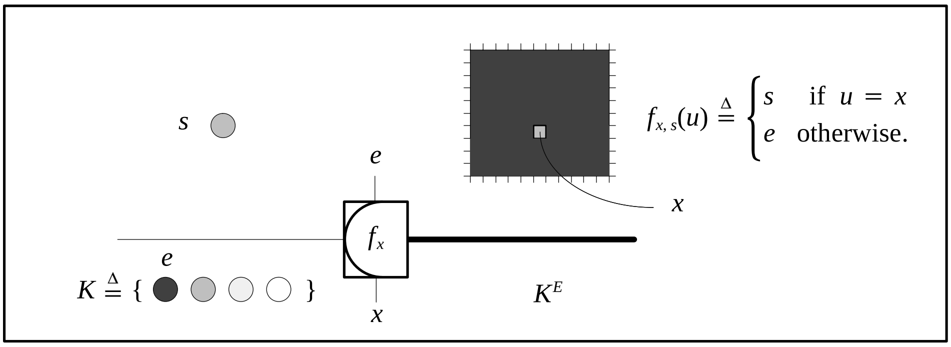


IMAGE PROCESSING OBJECTS

(3/4)

$f_x : \; K \rightarrow K^E$ (impulse image creator)



$\Psi : \; K_1^{E_1} \rightarrow K_2^{E_2}$ (operator)

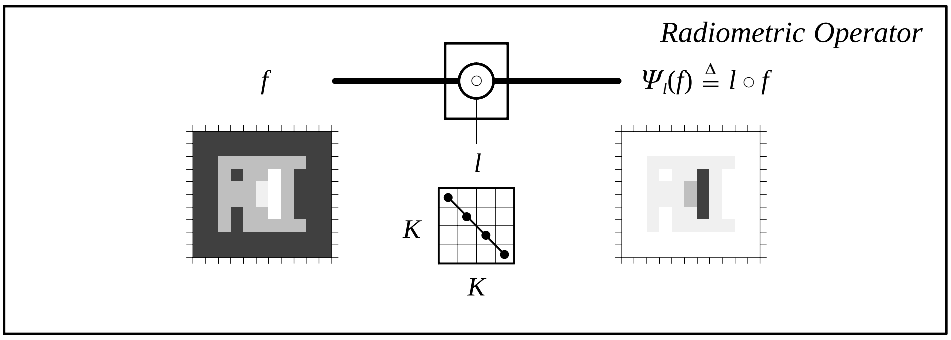
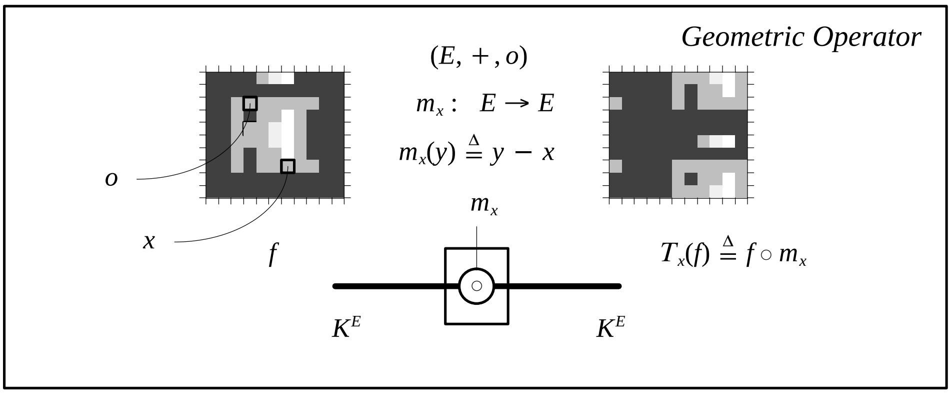


IMAGE PROCESSING OBJECTS

(4/4)



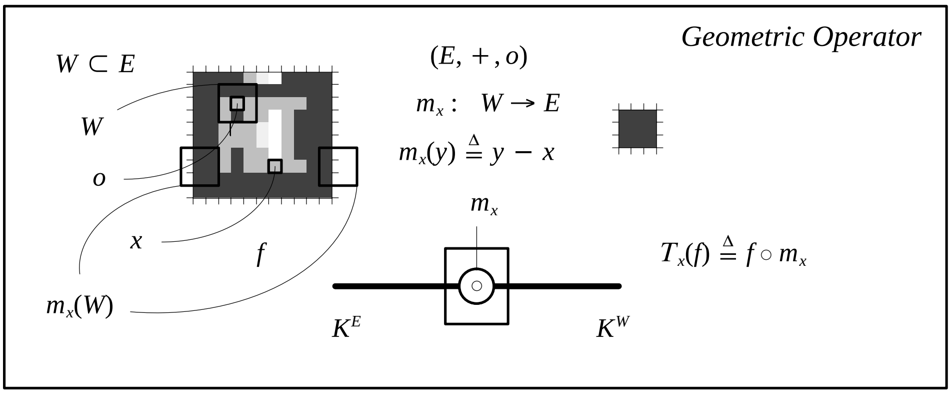


IMAGE OPERATIONS

(1/3)

Operations on gray–scale

$(K, \square, e)$ is a *commutative monoid* iff

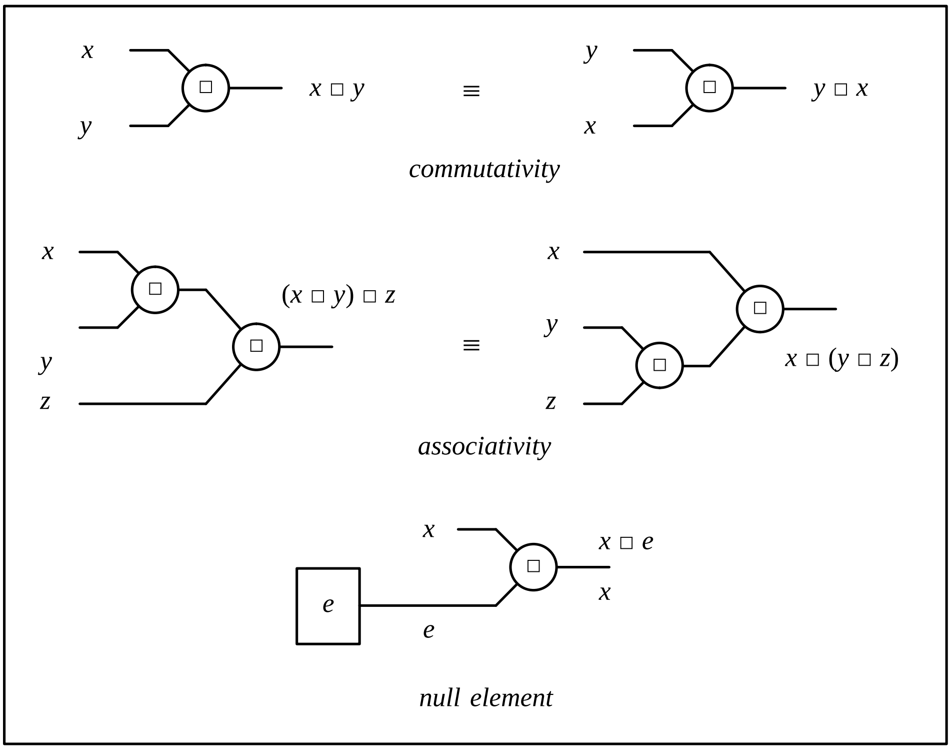


IMAGE OPERATIONS

(2/3)

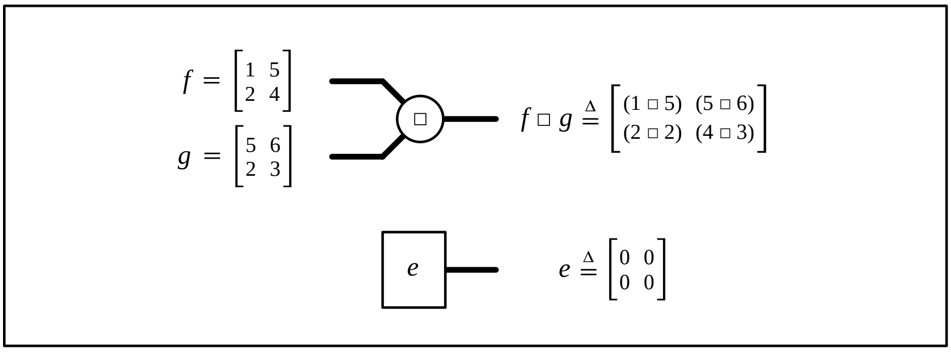
Let $K \triangleq \mathbf{R}$ (set of real numbers) then
$(K, +, \cdot, 0)$ is a **linear vector space** and
$(K, +, 0)$ is a **commutative monoid**.

Let $K \triangleq [0, k] \subset \mathbf{Z}$ and let
$s \vee t \triangleq \max\{s, t\}$ and $s \wedge t \triangleq \min\{s, t\}$, then
$(K, \vee, \wedge)$ is a **lattice** and
$(K, \vee, 0)$ and $(K, \wedge, k)$ are two **commutative monoids**.

IMAGE OPERATIONS

(3/3)

Operation extension to images



$(K^E, \square, e)$ is a **commutative monoid**.

Let $K \triangleq \mathbf{R}$, then
$(K^E, +, o)$ is a **commutative monoid**.

Let $K \triangleq [0, k] \subset \mathbf{Z}$, then
$(K^E, \vee, o)$ and $(K^E, \wedge, i)$ are **commutative monoids**.

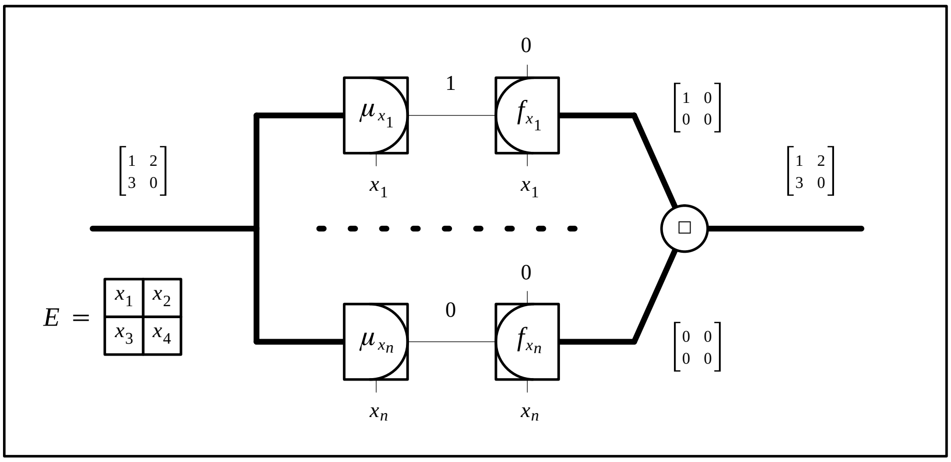
IMAGE DECOMPOSITION

(1/1)

Image decomposition
(consequence of $(K, \square, e)$ being a commutative monoid)

Let $E = \{x_1, x_2, ..., x_n\}$ and let $K = [0, k]$.
Example: $n = 4$ and $k = 4$.

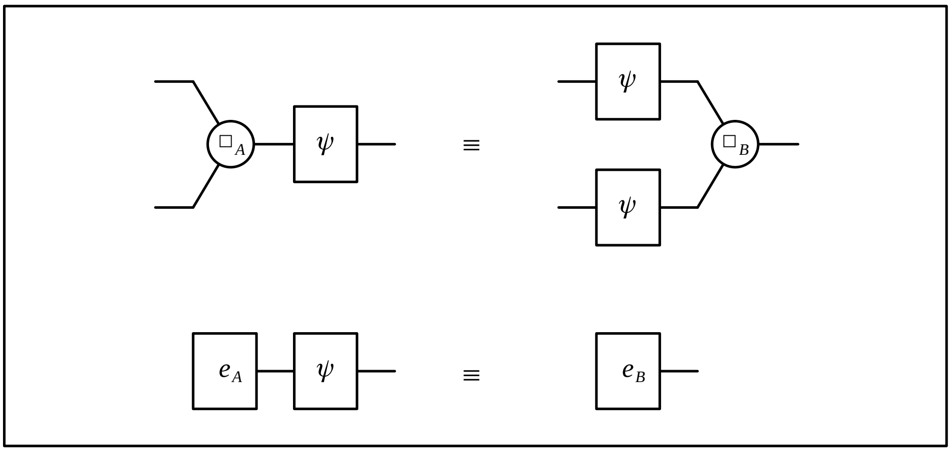
$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \square \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \square \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \square \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



MORPHISM OF COMMUTATIVE MONOIDS

(1/1)

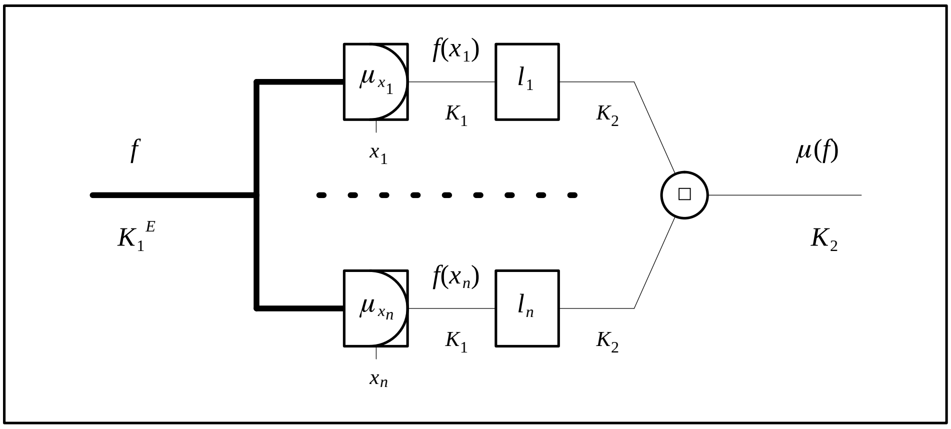
Let $(A, \square_A, e_A)$ and $(B, \square_B, e_B)$ be two commutative monoids.
A mapping ψ from A to B is a *morphism* iff



SEPARABLE MEASURES

(1/1)

Let $E = \{x_1, x_2, ..., x_n\}$.
A measure μ from ${K_1}^E$ to K_2 is *separable*
if there exists a family $\{l_1, l_2, ..., l_n\}$ of luts from K_1 to K_2
such that



Proposition (separability $\times$ morphism)
The measures which are morphisms of commutative monoids are separable and their characteristic luts are morphisms too.
(This is a consequence of the image decomposition.)

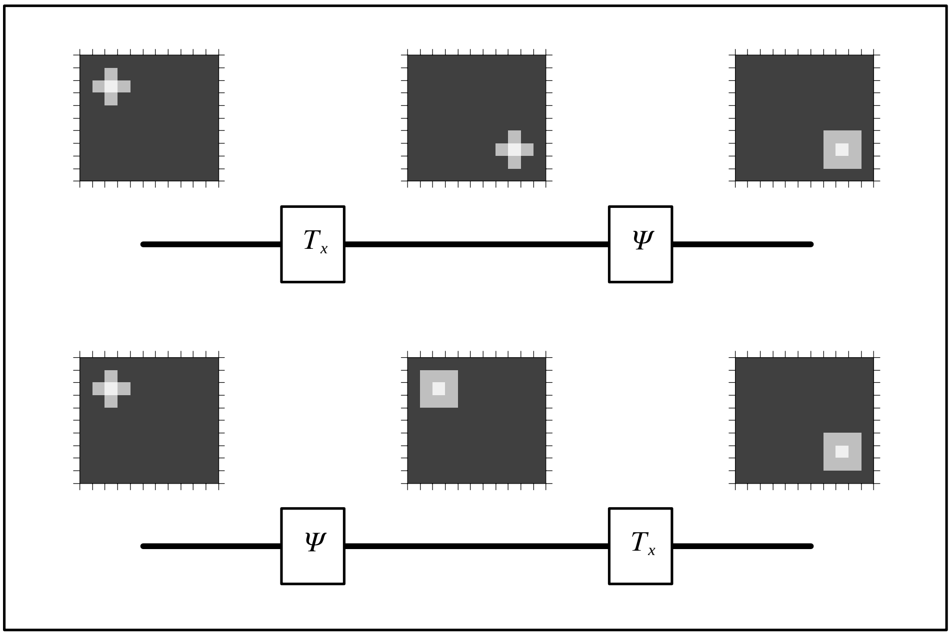
TRANSLATION INVARIANT OPERATORS

(1/1)

Let $(E, +, o)$ be an Abelian group.

Ψ is *translation invariant* (*ti–operator*) iff, for any x in E

$$T_x \circ \Psi = \Psi \circ T_x$$



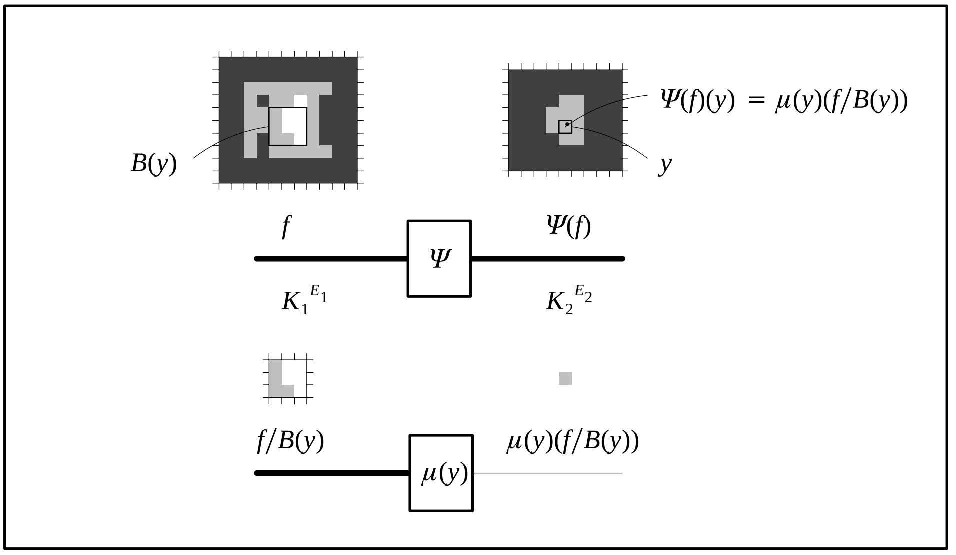
WINDOW OPERATORS

(1/1)

Let B be a mapping from E_2 to $\mathcal{P}(E_1)$.
Ψ is a *window operator* (*w–operator*) iff, for any y in E_2,

$$f/B(y) = g/B(y) \Rightarrow \Psi(f)(y) = \Psi(g)(y).$$

Let Ψ be w–operator,
then there exists a family $\mu(\cdot)$ of measures such that

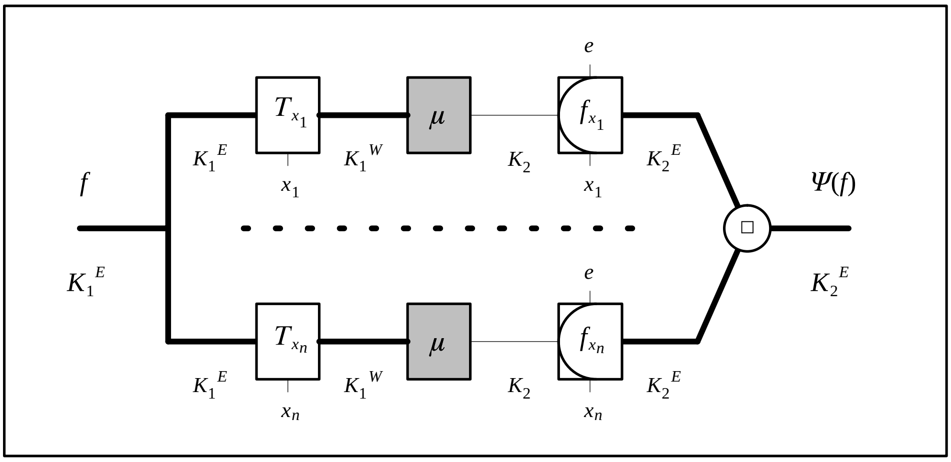


tiw–OPERATOR CHARACTERIZATION

(1/1)

Let $E = \{x_1, x_2, ..., x_n\}$.

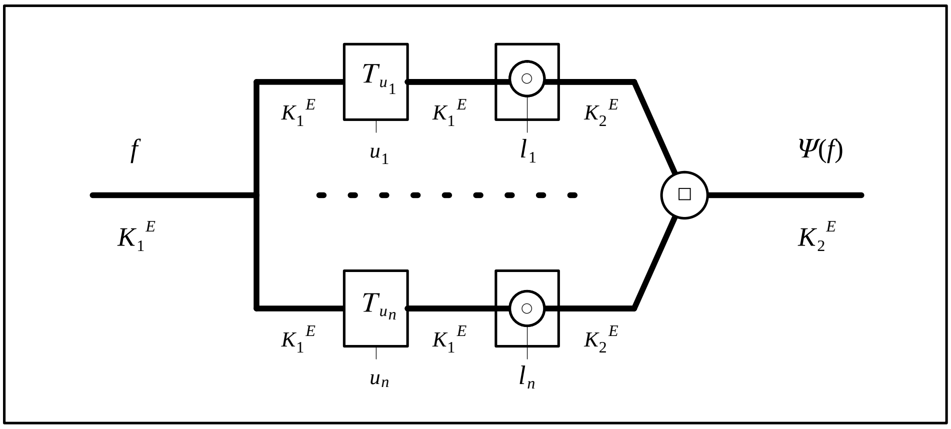
Let Ψ be tiw–operator, then there exists a measure μ such that.



tiw–MORPHISM CHARACTERIZATION

(1/1)

Let $W = \{u_1, u_2, ..., u_n\}$.
Let Ψ be tiw–morphism of commutative monoids,
then there exists a family $\{l_1, l_2, ..., l_n\}$ of luts
which are morphisms of commutative monoids
such that



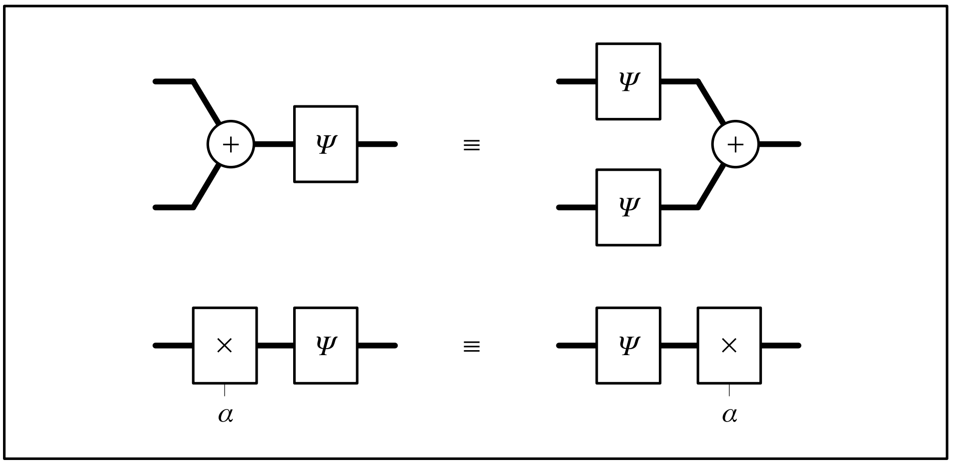
LINEAR OPERATOR CHARACTERIZATION

(1/3)

Let $K \stackrel{\Delta}{=} \mathbf{R}$, then
$(K^E, +, \cdot, o)$ is a **linear vector space**.

A *linear operator* is a morphism
from $(K^E, +, \cdot, o)$ to $(K^E, +, \cdot, o)$.
In other words, for any f, g in K^E and α in K,

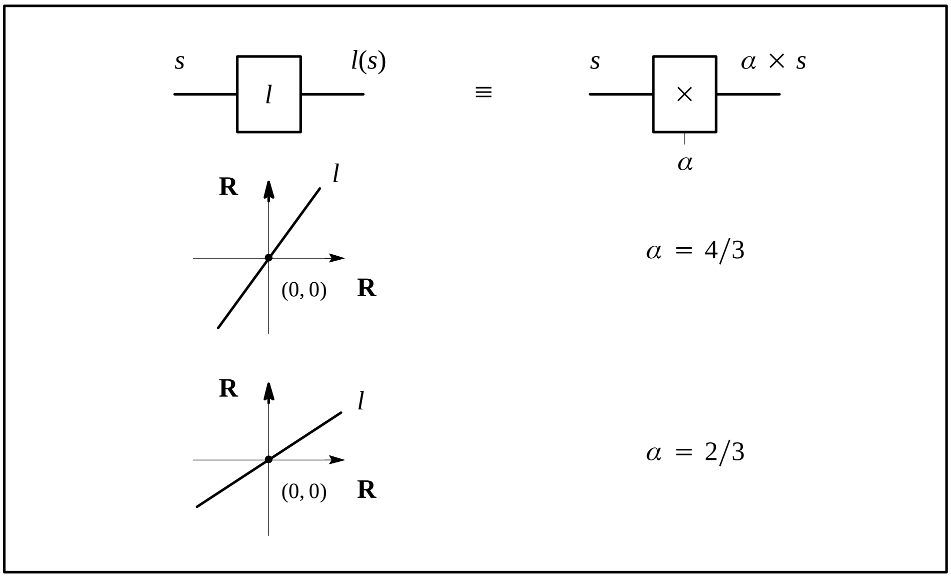
$$\Psi(f + g) = \Psi(f) + \Psi(g) \quad \text{and} \quad \Psi(\alpha \cdot f) = \alpha \cdot \Psi(f).$$



LINEAR OPERATOR CHARACTERIZATION

(2/3)

Let l be linear lut,
then there exists a real number α
such that

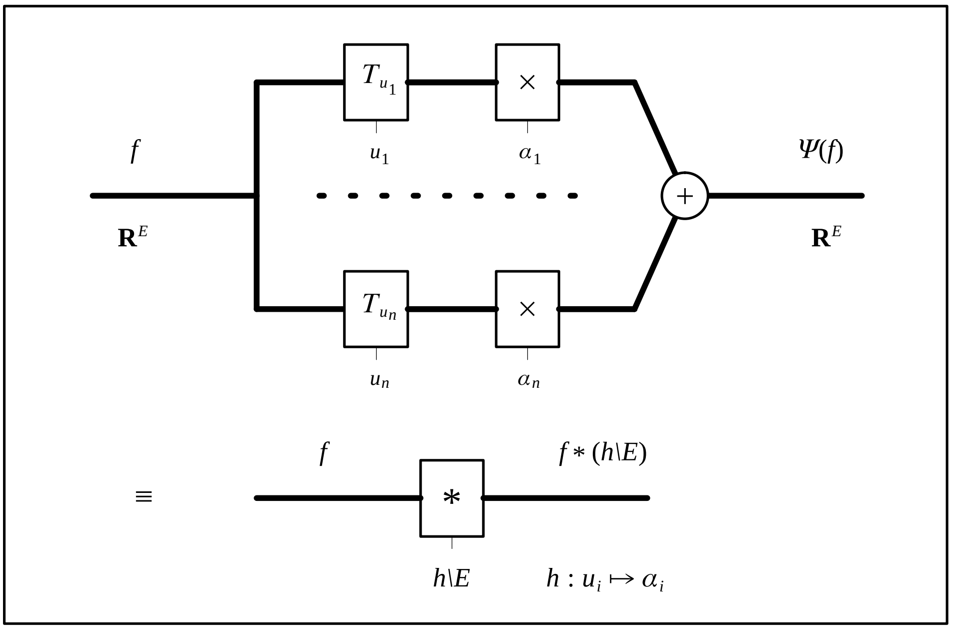


LINEAR OPERATOR CHARACTERIZATION

(3/3)

Let $W = \{u_1, u_2, ..., u_n\}$.

Let Ψ be linear tiw–operator on $\mathbf{R}^E$,
then there exists a family $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of real numbers such that



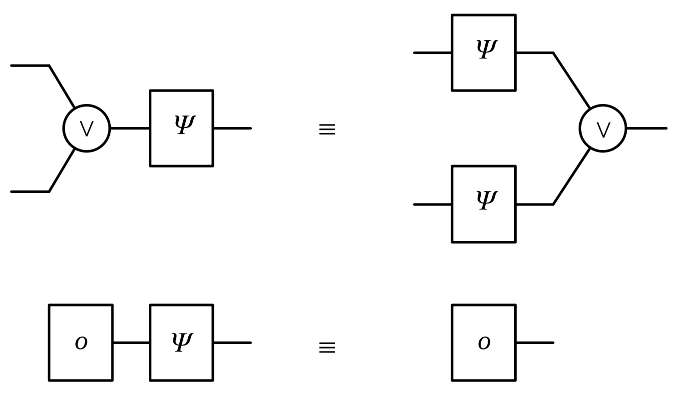
MORPHOLOGICAL OPERATOR CHARACTERIZATION

(1/7)

Let $K_1 \triangleq [0, k_1] \subset \mathbf{Z}$, $K_2 \triangleq [0, k_2] \subset \mathbf{Z}$ and
let $s \vee t \triangleq \max\{s, t\}$, then
$(K_1{}^E, \vee, o)$ and $(K_2{}^E, \vee, o)$ are two **commutative monoids**.

A *dilation* is a morphism
from $(K_1{}^E, \vee, o)$ to $(K_2{}^E, \vee, o)$.
In other words, for any f, g in $K_1{}^E$,

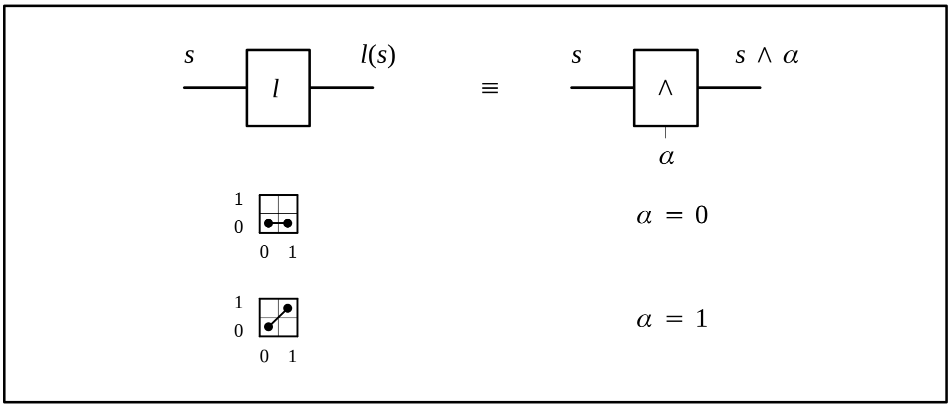
$$\Psi(f \vee g) = \Psi(f) \vee \Psi(g) \quad \text{and} \quad \Psi(o) = o.$$



MORPHOLOGICAL OPERATOR CHARACTERIZATION

(2/7)

Let l be a lut which is a dilation from $\{0,1\}$ to $\{0,1\}$, then there exists an integer number α in $\{0,1\}$ such that

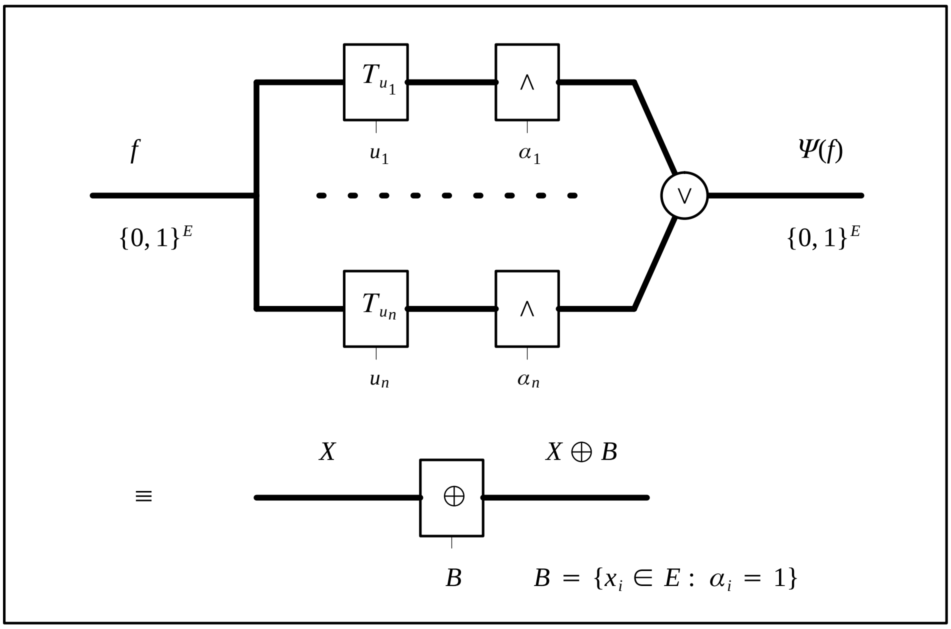


MORPHOLOGICAL OPERATOR CHARACTERIZATION

(3/7)

Let $W = \{u_1, u_2, ..., u_n\}$.

Let Ψ be tiw–operator which is a dilation on $\{0, 1\}^E$, then there exists a family $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of 0's and 1's such that

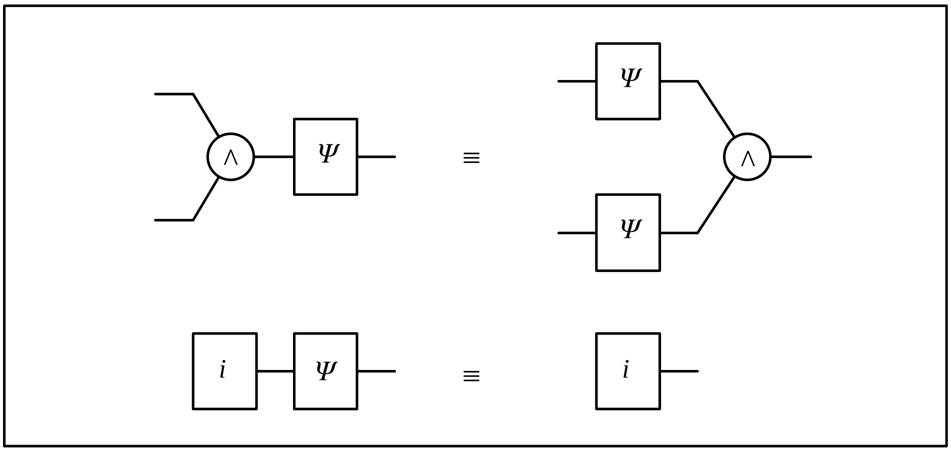


MORPHOLOGICAL OPERATOR CHARACTERIZATION
(4/7)

Let $K_1 \triangleq [0, k_1] \subset \mathbf{Z}$, $K_2 \triangleq [0, k_2] \subset \mathbf{Z}$ and
let $s \wedge t \triangleq \min\{s, t\}$, then
$({K_1}^E, \wedge, i)$, and $({K_2}^E, \wedge, i)$ are two **commutative monoids**.

An *erosion* is a morphism
from $({K_1}^E, \wedge, i)$ to $({K_2}^E, \wedge, i)$.
In other words, for any f, g in ${K_1}^E$,

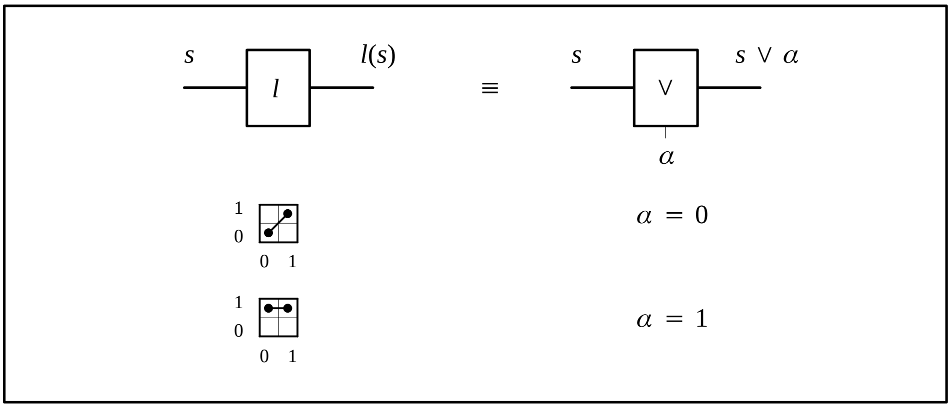
$$\Psi(f \wedge g) = \Psi(f) \wedge \Psi(g) \quad \text{and} \quad \Psi(i) = i.$$



MORPHOLOGICAL OPERATOR CHARACTERIZATION

(5/7)

Let l be a lut which is an erosion from $\{0,1\}$ to $\{0,1\}$,
then there exists an integer number α in $\{0,1\}$
such that

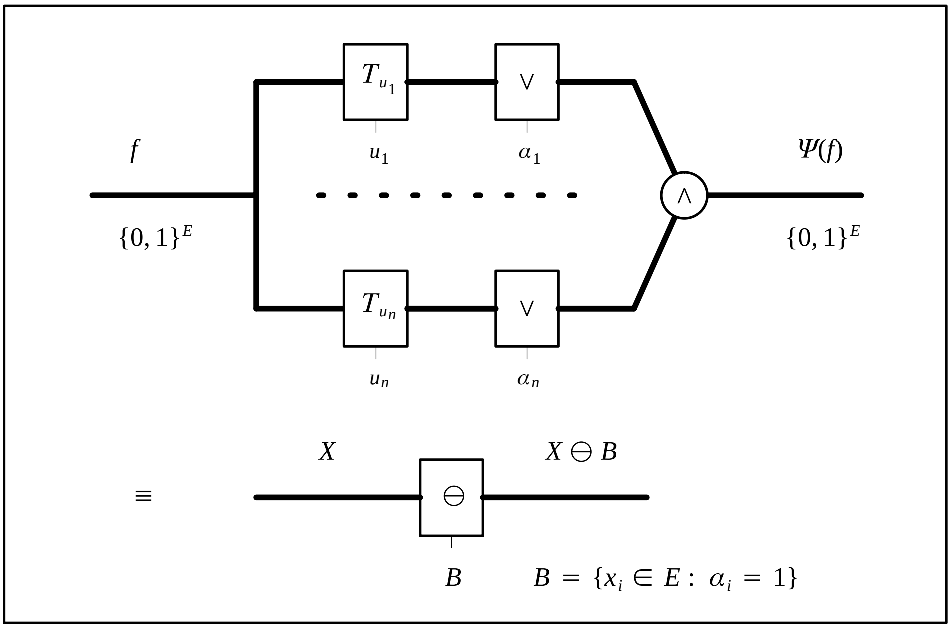


MORPHOLOGICAL OPERATOR CHARACTERIZATION

(6/7)

Let $W = \{u_1, u_2, ..., u_n\}$.

Let Ψ be tiw–operator which is an erosion on $\{0, 1\}^E$, then there exists a family $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of 0's and 1's such that

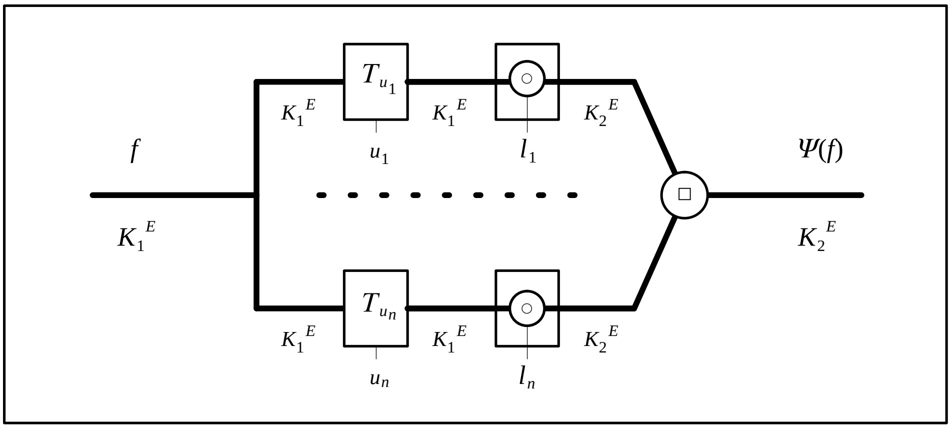


MORPHOLOGICAL OPERATOR CHARACTERIZATION

(7/7)

Non binary case

Let Ψ be tiw–operator which is a dilation (erosion) from $[0, k_1]^E$ to $[0, k_2]^E$, then there exists a family $\{l_1, l_2, ..., l_n\}$ of luts which are dilations (erosions) such that

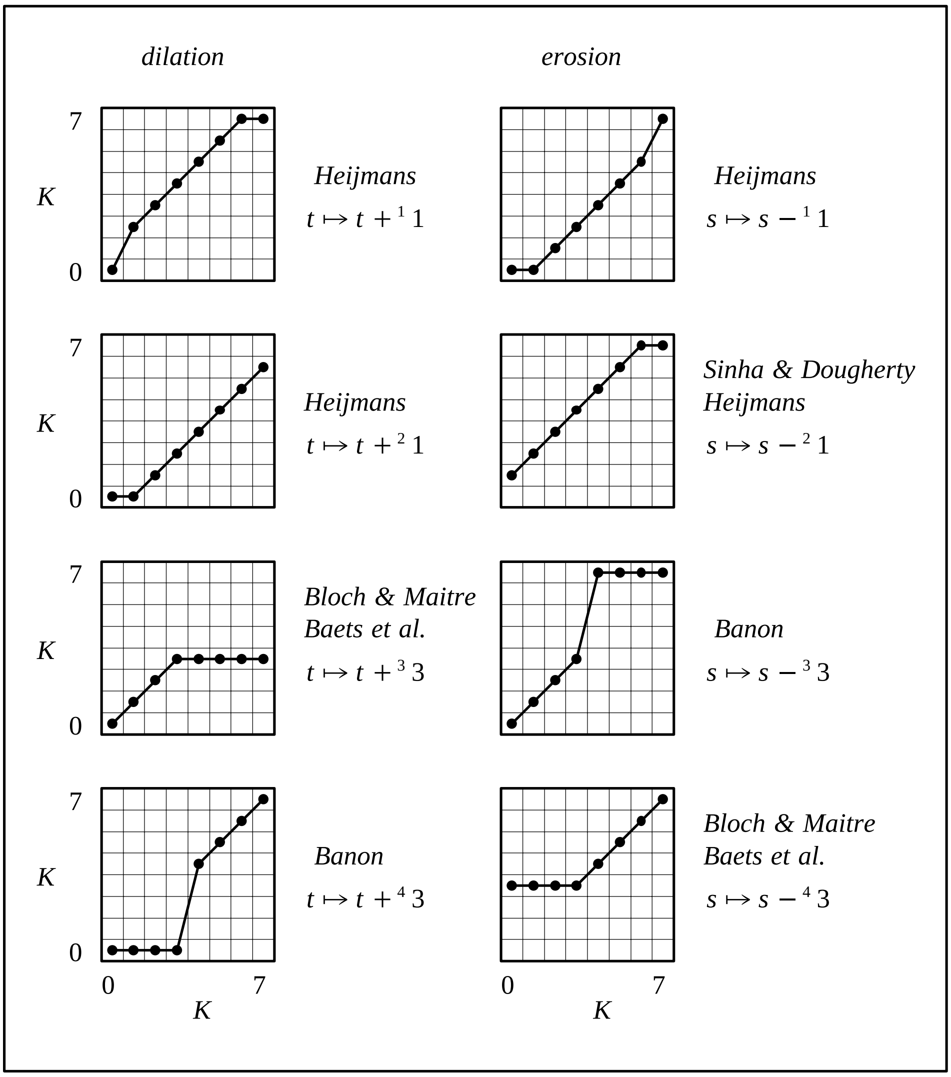


l is a dilation $\Leftrightarrow$ l is increasing and $l(0) = 0$

l is an erosion $\Leftrightarrow$ l is increasing and $l(k_1) = k_2$

Examples of morphological luts ($k_1 = k_2 = 7$)

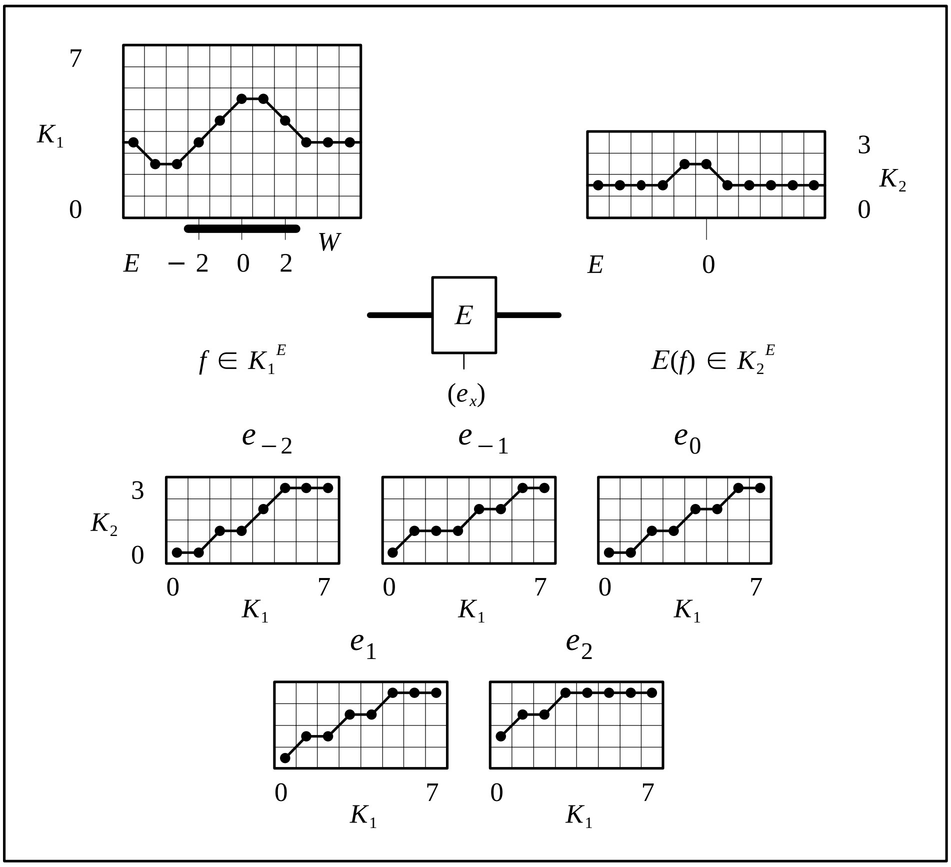
(see next transparency)


dilation
erosion
Heijmans
$t \mapsto t +^1 1$
Heijmans
$s \mapsto s -^1 1$
Heijmans
$t \mapsto t +^2 1$
Sinha & Dougherty
Heijmans
$s \mapsto s -^2 1$
Bloch & Maitre
Baets et al.
$t \mapsto t +^3 3$
Banon
$s \mapsto s -^3 3$
Banon
$t \mapsto t +^4 3$
Bloch & Maitre
Baets et al.
$s \mapsto s -^4 3$
K
0
7

EROSION EXAMPLE

(1/1)

Non binary case



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