Representation of Conics in the Oriented Projective Plane*

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Abstract. We present a geometric definition of conic sections in the oriented projective plane and describe some of their nice properties. This definition leads to a very simple and unambiguous representation for affine conics and conic arcs. A conic (of any type) is represented by the homogeneous coordinates of its foci and one point on it, hence, the metric plays a major role in this case as opposed to the traditional algebraic characterization of conics as second degree polynomial curves. This representation is particularly suitable for the implementation of geometric solutions of problems that involve the concept of distance. Furthermore, we discuss point location with respect to conic curves which constitutes an important elementary operation for the solution of many such problems.

1 Introduction

We find that among the various concerns in the process of designing data structures for representing geometric objects and the algorithms for manipulating them are dealing with numerical precision, various special cases, including ambiguity, and invariance under affine transformations. When the objects dealt with are limited to points, lines, line segments, and other equally simple ones, various techniques are available that can handle these difficulties either individually or globally. The situation is considerably more complicated for objects whose boundaries are described by curves. The coordinates of the endpoints are obviously not enough to represent an arc of a curve and additional information about the shape and position of the object is needed and in this case the literature is not so rich.

We concentrate here on the study of conics and conic arcs since these appear naturally in several applications including a large number of algorithms in computational geometry whose inputs are just points and straight line segments but whose output sometimes include all different types of conics.

Clearly, there are many ways to represent conics and conic arcs and the choice depends on the application. For instance, in one situation in computer graphics, I. Herman [1] presents a representation suitable for viewing pipelines, where the conics are described by a set of characteristic points and the goal is to achieve first affine and then projective invariance in order to reduce the cost of the pipeline process. From the characteristic points it is shown how to obtain a parametric equation so that arcs can be described by a subinterval of the parameter domain. Perhaps because of the requirements, each class of affine conics (i.e., ellipses, parabolas, and hyperbolas) has a different representation.

For geometric modeling applications, in [5] Farin shows how to represent conic arcs as NURBS curves. This representation is unique for all classes of conics, and is affine invariant. This is, in fact, a generalization of the known de Casteljau procedure to construct parabolic arcs.

In computational geometry, as far as we were able to determine, the work of M. Held [2] is the only one involving conics that discusses how to represent them analytically. In the problem of constructing offsets of boundaries formed by line segments and circular arcs, the conic arcs appear in the diagram used as a step to construct the offsets. Held uses a representation that is itself a parameterization of the conic arc. The parameter is the distance from the boundary. The representation is convenient for building offsets but is too specific to be used as a protocol in other applications.

Our Contribution. In this paper, we address the problem of representing conics and conic arcs in the context of computational geometry and propose a novel and simple representation which is particularly suitable for implementation of geometric algorithms. The majority of the problems in this field involves only straight objects. Nevertheless, there are algorithms whose output are planar subdivisions containing conic arcs. (We briefly review some of these algo-

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rithms in section 5.) Since these planar subdivisions are usually needed to allow for fast point location, we also show how to use the proposed representation for testing a point against the conic.

In general, text books on conic sections, such as [7], give more emphasis on the algebraic characterization than on the geometric one. Therefore, an ellipse is defined as a second degree equation whose coefficients satisfy certain constraints, instead of a set of points whose sum of distances from two given foci is constant. This is the right thing to do, as even in geometric *computations* we need a consistent mathematical semantic more than geometric intuition [3, p. 153]. However, as will be shown in section 4 this geometric definition, together with oriented projective geometry, leads to a simple representation that unifies all classes of affine conics.

The oriented projective plane $\mathbb{T}^2$ [3, 4] is the underlying space we use to emulate Euclidean plane and gain freedom to reason about conics in a simple way. This embedding has been called the two-sided Euclidean plane. For the convenience of the reader, we briefly review, in the next section, some of the basic concepts. Section 3 defines conics in $\mathbb{T}^2$. Section 4 presents the proposed representation for conics and conic arcs and discusses its use. Lastly, section 5 describes the occurrence and the use of conics in planar computational geometry.

2 Oriented Projective Geometry

Oriented projective geometry [3, 4] is a geometric model intended to serve as a better framework for geometric computations than the Euclidean or the classical projective geometries. It has the advantages of the projective space regarding the unification of concepts and simplification of computations, and furthermore allows the definition of orientation and convexity. These are among the most important concepts in which geometric computations are based on. We will present the ideas, and some notation, for the planar case. We assume that the reader is somewhat familiar with the use of (signed or unsigned) homogeneous coordinates.

In the oriented projective plane $\mathbb{T}^2$ a point with homogeneous coordinates $[x, y, w]$ is *not* identical to $[-x, -y, -w]$. They are called antipodal points. The antipode of point p is denoted by $\neg p$. While the classical projective plane $\mathbb{P}^2$ is the Euclidean plane $\mathbb{R}^2$ plus one line at infinity, $\mathbb{T}^2$ is composed by two copies of $\mathbb{R}^2$ plus one line at infinity. The set of points with $w > 0$ is called the front range (or side) of the plane, and those with $w < 0$ are the back range. The set of points with $w = 0$ (except for the invalid triplet $[0, 0, 0]$), which are referred to as improper points, is the line Ω at infinity. There are (at least) two geometric models for $\mathbb{T}^2$: the flat model, with the usual mapping $[x, y, w] \mapsto (x/w, y/w)$ to Cartesian coordinates, and the spherical model with the mapping $[x, y, w] \mapsto (x, y, w)/\sqrt{x^2 + y^2 + w^2}$. These two models are related by central projection (see Fig. 1(a)). Note that, in the flat model, antipodal points are coincident but can be thought of as being in opposite sides of the plane.

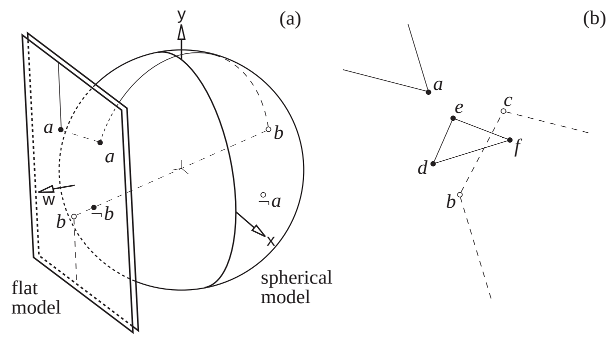

Figure 1: Models and conventions for $\mathbb{T}^2$

We represent elements in the front range by solid dots and lines, and open dots and dashed lines for the back range (see Fig. 1(b)).

The topology of $\mathbb{T}^2$ is determined by the spherical model. This makes $\mathbb{T}^2$ closely related to spherical geometry [9]. This relation will be important in section 3.

Lines and Segments. Lines in $\mathbb{T}^2$ have the same homogeneous equation of lines in $\mathbb{P}^2$, namely, $ax + by + cw = 0$. If we multiply this equation by -1 we get the oppositely oriented line $-ax - by - cw = 0$, which is composed by the same set of points. In the spherical model, lines are great circles on the sphere (see Fig. 2(b)). It should be noted that a line properly divides $\mathbb{T}^2$ into two subspaces, unlike $\mathbb{P}^2$, so that we can speak about the "left" and "right" sides of a line based on its orientation.

The segment between two points in $\mathbb{T}^2$ is defined as the set of points of the shortest arc of the great circle that contains the points. Note that, in the flat model, segments with endpoints in different sides of the plane have an unusual appearance (see segment ab in Fig. 1(b)).

The Join and Meet Operations. The join operation corresponds to the statement "the line defined by two points" and is denoted by the symbol $\vee$. Figure 2(a) shows the join $a \vee b$ which results in the line oriented from a to b. This figure also shows the left side of the resulting line, in the flat model. Note that in $\mathbb{T}^2$ two lines pass over two non-coincident points a and b, one oriented from a to b and the other from b to a.

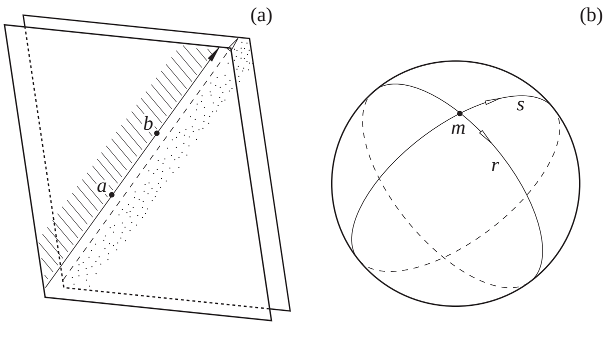

Figure 2: The join and meet operations

The meet operation corresponds to the statement "the point where two lines cross" and is denoted by the symbol $\wedge$. In $\mathbb{T}^2$ there are two (antipodal) points where two lines cross. Figure 2(b) shows how to decide which point is the meet of two given lines, considering the order that they are operated. If $m = r \wedge s$, and we move an arrow along r, agreeing with its orientation, m is the point where this arrow goes from the left to the right side of s.

Relative Position of Points and Lines. If we substitute the coordinates of a point p in the equation of a line r, we can decide in which side of r, p lies by looking at the resulting sign. This operation is denoted by the symbol $\diamond$:

$$p \text{ is} \left\{ \begin{array}{c} \text{to the left of} \\ \text{on} \\ \text{to the right of} \end{array} \right\} r \text{ if } p \diamond r = \left\{ \begin{array}{c} +1 \\ 0 \\ -1 \end{array} \right\}$$

2.1 Two-Sided Euclidean Plane

Problems in computational geometry are usually defined in $\mathbb{R}^2$, so that their solutions depend on perpendicularity, distance, and other Euclidean concepts. All these concepts can be defined in $\mathbb{T}^2$ if we assign a special meaning to the line at infinity Ω. Thus, we use $\mathbb{T}^2$ to emulate $\mathbb{R}^2$ [3, chapter 17]. This has been called the two-sided Euclidean plane. We believe that reasoning in this space can help the development and implementation of geometric algorithms in $\mathbb{R}^2$.

Perpendicularity. Let the line Ω be oriented so that the front range is its left side. Consider a proper line r. We say that $\operatorname{dir}(r) = r \wedge \Omega$, that is, the direction of r is the point where r crosses Ω from the front to the back range of the plane.

We say that two points a and b of $\mathbb{T}^2$ are polar to each other when their coordinates satisfy $a_x b_x + a_y b_y + a_w b_w = 0$, that is, in the spherical model, their corresponding vectors $(x, y, w)/\sqrt{x^2 + y^2 + w^2}$ are orthogonal. We call $\operatorname{norm}(r)$ the improper point polar to $\operatorname{dir}(r)$ and contained in the left side of r.

Two lines r and s are said to be perpendicular if $\operatorname{dir}(r) = \operatorname{norm}(s)$ or $\operatorname{dir}(s) = \operatorname{norm}(r)$ (see Fig. 3(b)).

Two-Sided Distance. The distance between two points may be defined straight from the usual Cartesian distance:

$$\operatorname{dist}(a, b) = \sqrt{\left(\frac{a_x}{a_w} - \frac{b_x}{b_w}\right)^2 + \left(\frac{a_y}{a_w} - \frac{b_y}{b_w}\right)^2}.$$

However, this formula does not distinguish antipodal points. We shall use the following signed expression:

$$\operatorname{dist}(a, b) = \frac{\sqrt{(a_x b_w - b_x a_w)^2 + (a_y b_w - b_y a_w)^2}}{a_w b_w}.$$

The latter formula yields negative numerical values when used with points in different sides of the plane, and positive values for points in the same side. Note that $\operatorname{dist}(a, b) = \operatorname{dist}(b, a)$, and that $\operatorname{dist}(\neg a, b) = \operatorname{dist}(a, \neg b) = -\operatorname{dist}(a, b)$. This unexpected behavior will be exploited in sections 3 and 4 where it will appear to be very convenient.

2.2 Relative Distance Between Proper and Improper Points

In $\mathbb{T}^2$ we can define without ambiguity *the* segment between a proper and an improper point, that is, what would be called a ray in $\mathbb{R}^2$. This allows the extension, by projective tools, of a very intuitive concept of $\mathbb{R}^2$: if we sweep the plane from infinity with a straight line, in a given direction, and the line encounters point b before point a, we can say that b is *closer to infinity in that direction* than a is.

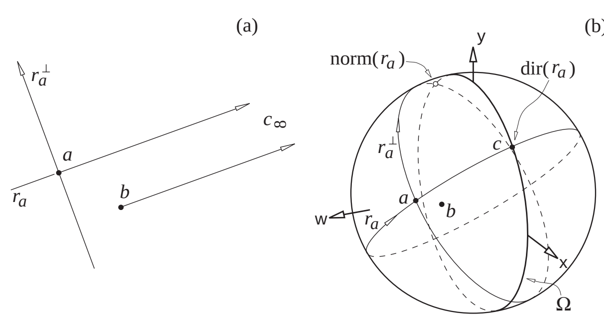

Figure 3: Relative distance from improper points

Definition 1 *Let a and b be two proper points and c an improper point* (Fig. 3(a). Note that segments ac and bc are parallel). *Let $r_a = a \vee c$ and $r_a^\perp = a \vee \mathrm{norm}(r_a)$. We shall say that:*

$$\left\{ \begin{array}{c} a \\ b \end{array} \right\} \textit{is closer to } c \textit{ than} \left\{ \begin{array}{c} b \\ a \end{array} \right\} \textit{if } b \diamond r_a^\perp = \left\{ \begin{array}{c} +1 \\ -1 \end{array} \right\}$$

and that a is as close to c as b if $b \diamond r_a^\perp = 0$.

We can also determine how much closer they are, simply by computing $\delta_r(b, r_a^\perp)$, where δ_r is the usual Euclidean distance from a point to a line. This concept is called *relative distance between proper and improper points*. Even though it may seem strange to compare $\mathrm{dist}(a, c)$ and $\mathrm{dist}(b, c)$ since both are infinite (and thus, might be regarded as equal), the comparison leads to no contradiction when we consider improper points in algorithms. This concept will help us establish the definition for conics in the next section.

3 Conics in Two-Sided Euclidean Plane

The usual relation between $\mathbb{R}^2$ and $\mathbb{P}^2$ is the mapping from Cartesian to homogeneous coordinates. Similarly, for $\mathbb{R}^2$ and $\mathbb{T}^2$. When transporting formulas and concepts from $\mathbb{R}^2$ to $\mathbb{T}^2$, the work is usually very simple: generally, all it takes is to substitute Cartesian for homogeneous coordinates. When dealing with polynomial equations, we should expect that when a point satisfies the equation, so will do its antipode. In the case of a straight line, the result of this algebraic approach matches the geometric definition for lines given by the axiomatic presentation of the spherical geometry. However, this is not the case with conics.

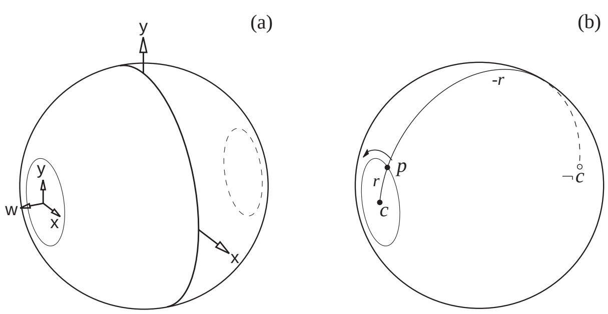

Figure 4: An algebraic circle in the two-sided plane and a geometric circle in the spherical geometry

We will first discuss the simpler case of the circle, and later generalize the discussion to the other conics.

Consider the circle $x^2 + y^2 - 1 = 0$. In homogeneous coordinates we have $x^2 + y^2 - w^2 = 0$. The set of points which satisfy this equation has two connected components (Fig. 4(a)), which is an undesirable situation. We can not properly define *the* center of this circle unless we identify antipodal points which would lead us to $\mathbb{P}^2$ and its unorientable topology, which we want to avoid. Instead, we will use the geometric definition for circles in the spherical geometry [9]. A circle is a set of points whose distance from a center c equals a radius r (Fig. 4(b)) — note the explicit reference to *distance*. This is the same set of points of the circle centered at $\neg c$ and radius $-r$. To distinguish these coincident circles, we can assign opposite orientations to them. If we move a point p along a circle in counterclockwise direction, as seen from the outside of the sphere, the segment cp rotates counterclockwise around c, but $\neg cp$ rotates clockwise around $\neg c$. Finally, we will say that those two connected components of the algebraic circle are, actually, antipodal circles, each one with two possible orientations.

To define the other types of conics we pursue the same approach. In order to help the reader's intuition, Fig. 5(a) shows the two connected components of a hyperbola and Fig. 5(b) shows a pair of parabolas (each with two connected components), $yw = x^2$ and $(y - 100w)w = (x - 100w)^2$, which differ by a translation of $[100, 100, 1]$[1]. Notice that the parabolas are tangent to Ω at two antipodal points, $[0, 1, 0]$ and $[0, -1, 0]$, which are the intersections of Ω with their focal lines. Later in this section, we will see how to distinguish between the two connected components of these conics.

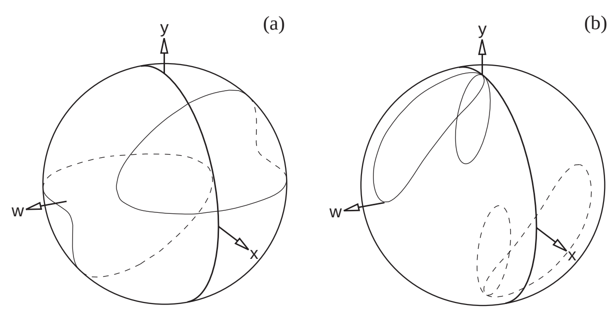

Figure 5: Algebraic conics in the two-sided plane

In spherical geometry, conics can be geometrically defined as the set of points whose sum of distances from two given foci is constant [8, chapter X]. This is the definition for ellipses in $\mathbb{R}^2$, and in fact, sphero-conics "look" like ellipses. They are, algebraically, the intersection of the sphere with a

[1] We used the *sphereView* visualizer of the *GeoPrO* environment [10] to simulate these conics from the point of view of this figure.

second degree cone having its vertex in the center of the sphere. The two foci of the sphero-conic are the points where the focal lines of the cone intersect the sphere. The intersection of such cones with planes are second degree curves, namely affine conics in the plane. Thus, we see that the images of affine conics in the spherical model of $\mathbb{T}^2$ (see Fig. 5) are, indeed, sphero-conics.

It is easy to see that the images of the foci of the affine conic are *not* the foci of the projected sphero-conic. Despite this, we can also apply the same definition with respect to the affine foci in the plane. To see this, imagine a dynamic scenario with projective maps. It is known that a parabola may be considered, in every aspect, as an ellipse with one of its foci moved to infinity [7, page 202]. In the two-sided plane we can push this focus beyond infinity and consider the resulting hyperbola as an ellipse with foci in different sides of the plane.

Definition 2 *In the two-sided Euclidean plane, a conic is a set of points whose* sum *of distances from two given foci is constant.*

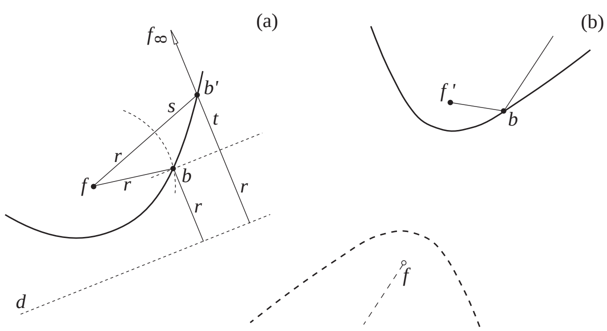

Figure 6: The definition 2 as seen from affine conics

This definition yields an ellipse if the foci are in the same range, a parabola if one focus is at infinity and a hyperbola if they are in different ranges. We need to show how the affine parabola and hyperbola fit this definition in the flat model. For the parabola we use the ideas of section 2.2. Consider the parabola in Fig. 6(a). As we move from b to b' the relative distance from the improper focus decreases by t. From the definition of affine parabolas we know that $r+t = r+s$, so that $s = t$ and the distance from the proper focus increases by t. For the hyperbola, note that one of the distances is always negative. In Fig. 6(b) we have $\mathrm{dist}(b,f) = -\mathrm{dist}(b,\neg f)$. Therefore, if the foci are in different sides of the plane, the effect of adding the two distances is to subtract their absolute values and we get a hyperbola.

Now, as in the case of circles, we consider the two connected components of the algebraic conic as antipodal conics. For instance, consider the conic with foci f and f', and the constant sum c. If p is a generic point, the equation is $\mathrm{dist}(p,f)+\mathrm{dist}(p,f')= c$. To get the opposite oriented conic we multiply the equation by -1: $-\mathrm{dist}(p,f)-\mathrm{dist}(p,f')= -c$, which is the same as $\mathrm{dist}(p,\neg f)+\mathrm{dist}(p,\neg f')= -c$. To get the antipodal conic we change the sign of the constant sum: $\mathrm{dist}(p,\neg f)+\mathrm{dist}(p,\neg f')= c$. To change the orientation of this latter conic we multiply by -1: $\mathrm{dist}(p,f)+\mathrm{dist}(p,f')= -c$. Note that if we try to extract the radicals, by appropriately squaring these equations twice, we get the polynomial equation of the algebraic conic which does not distinguish antipodal points. We will discuss this issue in section 5. Before proposing the computational representation, we list three properties of the geometric conic given by definition 2:

1. If we remove a conic from $\mathbb{P}^2$ what remains are two subspaces, one of them topologically equivalent to an open disc and the other equivalent to a Möbius strip. If we remove a conic from $\mathbb{T}^2$ what remains are two subspaces, both equivalent to an open disc.

2. Let us call the interior of a conic *the* subspace which contains at least one of the foci. So, the segments joining *any* point on a conic to its foci are entirely contained in the interior of the conic, as opposed to what happens on hyperbolas in $\mathbb{R}^2$.

3. From the preceding property we see that the interior of a conic is always star-shaped with respect to the foci. Furthermore, if the conic is oriented counterclockwise its interior is convex. A conic is oriented counterclockwise if we move a point on it, in counterclockwise direction as seen from the outside of the sphere, and the segments joining this point to the foci rotate counterclockwise around them.

4 Representation for Conics and Conic arcs

We represent a conic by the homogeneous coordinates of three points in the two-sided Euclidean plane. The points are the two foci f_1 and f_2, and one point d on the conic. By definition 2, this set of points uniquely defines the conic. The affine class of the conic is implicitly given by the location of the foci. For all classes, to change the orientation of the conic, we simply exchange the foci for their antipodes, which means multiplying their coordinates by -1. To get the antipodal conic, we exchange the point d for its antipode.

Figure 7 shows, in the flat model, the parabolas of Fig. 5(b). Their two connected components are now different parabolas and have different representations. In Fig. 7(a), $f_1 = [0, 0.25, 1]$, $f_2 = [0, 1, 0]$, $d = [0, 0, 1]$, $f_1' = [100, 100.25, 1]$, $f_2' = f_2$, and $d' = [100, 100, 1]$. Whereas in Fig. 7(b), $d = [0, 0, -1]$ and $d' = [-100, -100, -1]$.

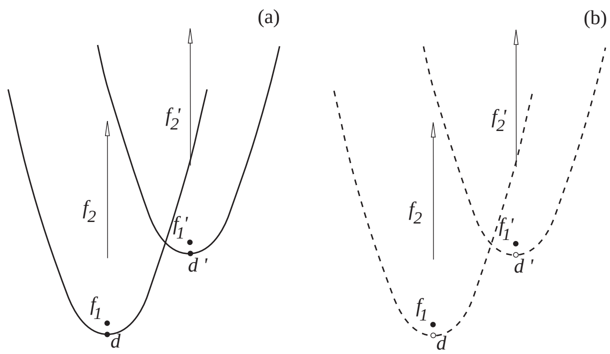

Figure 7: Representation of parabolas in the flat model

Representation for Arcs. The coordinates of the endpoints of an arc may, themselves, be used to determine the shape of the conic. This is the case in the work of G. Farin [5]. However, as will be shown in the next section, the coordinates of the foci of the conic are usually present in the input of algorithms. This means that we better constrain the representation of the conic to use these coordinates as its foci directly, in order to minimize errors. Thus, we represent an arc of a conic with the same three points (f_1, f_2 and d to represent the conic itself) and two more points a_1 and a_2 to help determine the endpoints of the arc.

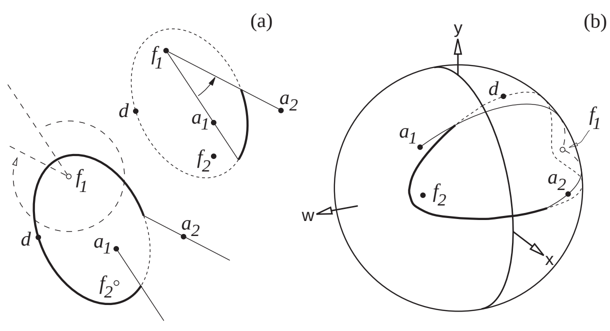

Figure 8: Representing arcs in the two-sided Euclidean plane

Due to the fact that the conic is star-shaped with respect to the foci, all lines passing through one focus intersect the conic in two points. Let $r_1 = f_1 \vee a_1$ and $r_2 = f_1 \vee a_2$. We define the starting point of the arc as the intersection between the conic and r_1 when it leaves the interior of the conic (see Fig. 8(b) for an arc of a hyperbola). Let this intersection be called the first endpoint. The second endpoint is, as expected, the intersection between the conic and r_2 when it leaves the interior of the conic. Note that there are two complementary arcs satisfying this representation. We use the orientation of the conic to decide which arc is represented. The arc is the one traced by the first intersection when we rotate r_1 counterclockwise around f_1 until r_1 coincides with r_2. For all classes, to exchange an arc for its complement, either change the orientation of the conic (see Fig. 8(a)) or exchange a_1 for a_2. To get the antipodal arc, exchange d, a_1 and a_2 for their antipodes.

Degenerate Cases. This representation can describe without redundancy any arc of non-degenerate conics. However, one may set the coordinates of the points f_1, f_2, d, a_1 and a_2 so that the resulting data does not properly define a conic. We identify these degenerate situations:

1. When a_1 (a_2) and f_1 are coincident (equal or antipodal), r_1 (r_2) is not defined.
2. When both f_1 and f_2 are at infinity.
3. When f_1 and f_2 are antipodal points.
4. When d is at infinity. We observe that if f_1 and f_2 are in different sides of the plane, the hyperbola is indeed well defined. In the other cases, we should regard the conic to be equal to Ω. However, all these cases would require special computational treatment.

An application, in order to be robust, must know about all these degenerate cases that may need a special treatment or even an exception. We should emphasize that there is no arc of non-degenerate conic that cannot be represented in the proposed way.

Before discussing the use of the representation we also observe that it is *not* affine invariant, as it relies explicitly on distances. However, it is indeed similarity invariant (e.g. translations, rotations, and uniform scaling).

4.1 Testing points against a conic

The two-sided distance has the following behavior when we move a point away from another fixed point: the distance varies in $[0, +\infty]$, and then in $(-\infty, -0]$. We refer the reader to [3] for a formal interpretation of these values. To perform point location with the proposed representation it is enough to compare the sum of distances with respect to that (unusual) domain.

When an ellipse is oriented counterclockwise, $\mathrm{dist}(d, f_1) + \mathrm{dist}(d, f_2)$ is a positive constant c. Consider a query point q. If q is *not* in the same range of d, it is in the exterior of the ellipse and no computation is needed. If this is not the case, set $c' = \mathrm{dist}(q, f_1) + \mathrm{dist}(q, f_2)$. q is in the interior of the ellipse if c' is between 0 and c, on the ellipse if $c' = c$, and in the exterior otherwise. For the clockwise oriented ellipse c is a negative constant. The point q is in the exterior if c' is between c and 0.

When a hyperbola is oriented counterclockwise, $\mathrm{dist}(d, f_1) + \mathrm{dist}(d, f_2)$ is a negative constant c. The point q is in the interior of the hyperbola if c' is between $-\infty$ and c. For the clockwise oriented hyperbola c is a positive constant. The point q is in the exterior if c' is between c and $+\infty$.

For the parabola we must use the relative distance from the improper focus f_i. When the parabola is oriented counterclockwise, $\mathrm{dist}(d, f_p)$ is a positive constant c, where f_p is the proper focus. If q is *not* in the same range of d, it is in the exterior of the parabola and no computation is needed. If this is not the case, let $r_f = f_p \vee f_i$ and $r_d = d \vee \mathrm{norm}(r_f)$. If $q \diamond r_d = -1$, q is closer to f_i than d, so set $c' = \mathrm{dist}(q, f_p) - \delta_r(q, r_d)$. If $q \diamond r_d = +1$, set $c' = \mathrm{dist}(q, f_p) + \delta_r(q, r_d)$. Finally, q is in the interior of the parabola if c' is between 0 and c, on it if $c' = c$, and in the exterior otherwise.

For the clockwise oriented parabola, c is a negative constant. When q is in the same range of d, if $q \diamond r_d = -1$, d is closer to f_i than q, so set $c' = \mathrm{dist}(q, f_p) + \delta_r(q, r_d)$. If $q \diamond r_d = +1$, set $c' = \mathrm{dist}(q, f_p) - \delta_r(q, r_d)$. Then, q is in the exterior of the parabola if c' is between c and 0.

Testing against arcs. For conic arcs we observe that lines r_1 and r_2 are all we need to compose the test together with the previous ones. These lines define the sector of the conic which contains the arc. So, besides being in the interior of the conic, the query point q must be inside this sector. If point a_2 is to the left of r_1, q must be to the left of r_1 *and* to the right of r_2. If the point a_2 is to the right of r_1, q must be to the left of r_1 *or* to the right of r_2.

This sector can be used to decide the relative position of q in the neighborhood of the arc.

5 Applications in Planar Computational Geometry

Geometric conics arise naturally in problems involving distance. The most studied are the so called generalizations of the Voronoi diagram. Consider a set of points, called sites, in the plane. Voronoi diagrams are, conceptually, the partition of the plane into regions, each one associated to one site, such that all points inside a given region are closer to the corresponding site than to all other sites. These regions are bounded by line segments. The generalizations are obtained when we have more complex objects as sites, or when we change the meaning of "closer" varying the underlying metric. We discuss, in more detail, the Voronoi diagram of additively weighted points [11].

The additively weighted points Voronoi diagram is obtained when each site has an associated weight. The distance between an ordinary point p and a site s is given by $\delta(p, s) - w(s)$, where δ is the usual Euclidean distance and $w(s)$ the weight of s. The regions are bounded by hyperbolic arcs having the sites as foci (see Fig. 9(a), the numbers indicate the weight of the sites).

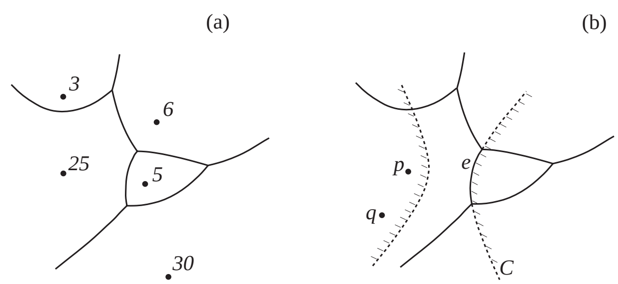

Figure 9: Voronoi diagram of weighted points

We consider the hyperbolas of this diagram as being of the geometric kind. Only one branch of the algebraic hyperbola is present. Note that the only way to decide if a given query point q is inside the region of site p, in the neighborhood of edge e, is to test it against the hyperbola C (see Fig. 9(b)). If the second degree polynomial equation is used, the two branches of the hyperbola are not distinguished, so that we must do some further test to avoid errors.

Many other Voronoi diagrams have conic arcs. The diagram of line segments [11] has parabolic arcs. The geodesic Voronoi diagram of points inside a polygon [12] has hyperbolic arcs. In [13] the Voronoi diagram of a set of points, line segments, and circular arcs is considered. This diagram has elliptic, parabolic, and hyperbolic arcs all together.

Besides Voronoi diagrams, geometric conic arcs appear, at least, in visibility diagrams and shortest-path maps. The common point between all these problems is the concept of distance in their definitions, and the presence of the foci of the conics in the input.

6 Conclusion

We have presented a geometric definition for conics in the two-sided Euclidean plane that allow us to reason about all classes of affine conics in a unified way. Straight from the definition, a simple computational representation for conics and conic arcs was proposed. This representation explicitly uses the sign of the homogeneous coordinates to characterize the conic. We also discussed point location with respect to the conics. As this representation is based on the concept of distance, it is suitable for the planar subdivisions which appear in many algorithms in computational geometry. Even if the computation is done in the Euclidean plane, the operations with conics can be simplified with this representation.

This representation was implemented as a protocol for conics and conic arcs in the GeoPrO environment for distributed visualization [10]. We are also using it to construct generalizations of the Voronoi diagram. We observe that rendering conic arcs from this representation is straightforward. We can use the parameterization based on the eccentricity and directrix of the affine conic [6]. The interval of the domain of parameterization is given by the angles that lines r_1 and r_2 make with the focal line of the conic ($f_1 \vee f_2$).

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