The Size-Intensity Diagram: a Gray-scale Granulometric Analysis Tool

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Abstract. We present a Size-Intensity Diagram, based on a gray-scale granulometry. For this we use a cylindrical structuring function with two varying parameters: the radius r and the height (or intensity) k. The measurement associated with this gray-scale granulometry is based on the volume measurement commonly used in the classical gray-scale granulometry. The proposed diagram is therefore a bidimensional space of variables r and k. The variation of both r and k provides the integration of the size and intensity information into a single graphic. This representation can be seen both as adding size information into the histogram and as adding intensity distribution information to the granulometry curve.

1 Introduction

Among the problems of image analysis addressed by Mathematical Morphology, the extraction and quantization of shape-size information are the ones that take better advantage of the subjective concept of shape implied in Morphology. Granulometries have been used as such a tool and its use for binary images is well established and applied [Ser82].

Another domain of interest in the field of image processing is the determination of intensity distribution in an image. Statistical operators like the classical intensity histogram are often applied to characterize the intensity distribution of an image.

Granulometry tools can also be applied to gray-scale images and although they have been used by many authors, we think their use has not reached the same maturity as the binary granulometry tools. Classically, gray-scale granulometries have been used with 3-D semi-spherical or pyramidal structuring functions and measurements taken as the volume of the umbra of the image [Mar89].

We can show that the classical intensity histogram is a gray-scale granulometric curve when using sequences of openings by structuring functions composed by a single pixel.

The above considerations have led us to propose a new, more generic, tool for granulometric analysis, the so-called *Size-Intensity Diagram – SID* [Lot96]. In this paper we redefine some aspects concerning the definition of the SID, in particular the choice of the measurement function, which is now taken as the volume of the opened image, instead of the area originally proposed, and go further in the analysis of the information provided by such a diagram, showing how the Pattern Spectrum and Intensity Histogram can be derived as particular cases of the SID. We also illustrated a situation which demonstrates the richness of the SID compared to the classical granulometry.

The paper is organized as follows: in Section 2 we review the notation and definitions of the basic mathematical morphology operators. Section 3 makes a revision of granulometry concepts. Section 4 describes the conception of the SID, with a few enhancements. In Section 5 we present results of the use of this granulometry in synthetic and real images and discuss the interpretation of such data. Finally, Section 6 gives conclusions and directions for future work.

2 Basic Definitions

Let $\mathbf{Z}$ be the set of integers, let E be a rectangle on $\mathbf{Z}^2$ representing a subset of the square grid and let K be an interval $[0,k]$ of $\mathbf{Z}$ with $k > 0$. The collection of functions from E to K will represent the *gray-scale* images of interest. We denote such a collection by K^E and by f a generic element of K^E.

Let B be a subset of $\mathbf{Z}^2$ called the *structuring element*. We denote by $B + h$ the *translate* of B by any vector h in $\mathbf{Z}^2$:

$$B + h = \{x + h | x \in B\} \quad (1)$$

We denote by B^t the *transpose* of B, that is,

$$B^t = \{-x | x \in B\} \quad (2)$$

Let g be a *structuring function* defined from $B \subset E \oplus E^t$, the *dilation* of f by g is the function $\delta_g(f)$ in K^E, given by, for any x in E,

$$\delta_g(f)(x) = max\{f(y) \dot{+} g(y-x) | y \in (B^t + x) \cap E\} \quad (3)$$

and the *erosion* of f by g is the function $\varepsilon_g(f)$ in K^E, given by, for any x in E,

$$\varepsilon_g(f)(x) = min\{f(y) \dot{-} g(y-x) | y \in (B+x) \cap E\} \quad (4)$$

In the above formulation the dotted addition and subtraction, $\dot{+}$ and $\dot{-}$, are operations from $K \times \mathbf{Z}$ to K defined, for any $t \in K$ and $v \in \mathbf{Z}$, by:

$$t \dot{+} v = \begin{cases} 0 & \text{if } t = 0, \\ 0 & \text{if } t > 0 \text{ and } t + v \leq 0, \\ t + v & \text{if } t > 0 \text{ and } 0 \leq t + v \leq k, \\ k & \text{if } t > 0 \text{ and } t + v > k \end{cases} \quad (5)$$

and

$$t \dot{-} v = \begin{cases} 0 & \text{if } t < k \text{ and } t - v \leq 0, \\ t - v & \text{if } t < k \text{ and } 0 \leq t - v \leq k, \\ k & \text{if } t < k \text{ and } t - v > k, \\ k & \text{if } t = k \end{cases} \quad (6)$$

The morphological opening is given by an erosion followed by a dilation:

$$\gamma_g(f) = \delta_g \varepsilon_g(f) \quad (7)$$

3 Granulometry

The concept of granulometry, as stated by G. Matheron [Mat75], is based on the following definition:

Definition 1 *Let $\Phi = (\phi_\lambda)_{\lambda > 0}$ be a family of image transformations depending on a unique parameter λ. This family constitutes a granulometry if and only if the following properties are satisfied:*

$$\begin{gathered} \forall \lambda \geq 0, A \subset B \Rightarrow \phi_\lambda(A) \subset \phi_\lambda(B), \\ (\phi_\lambda \textit{ is increasing}) \\ \forall \lambda \geq 0, \phi_\lambda(A) \subset A, \\ (\phi_\lambda \textit{ is anti-extensive}) \\ \forall \lambda, \mu \geq 0, \phi_\lambda[\phi_\mu(A)] = \phi_\mu[\phi_\lambda(A)] = \\ \phi_{max(\lambda,\mu)}(A), \textit{(the stronger sieve)} \end{gathered} \quad (8)$$

It can be shown that these three properties, known as *Matheron axioms* are satisfied by the definition of morphological openings. Consequently, a family of openings $[\Phi_g(f) = \gamma_g(f)]$, based on any convex set B, constitutes a granulometry.

If we define a function m that measures $\Phi_g(f)$ in some way (area for binary images, volume for gray-scale, etc.), the histogram generated by increasing g can be viewed as a *signature* of f for that particular choice of g, which thus provides a granulometric curve, often referred to as the *Pattern Spectrum* of f, $PS_\Phi(f)$:

$$PS_\phi(f)(n) = m(\phi_n(f)) - m(\phi_{n-1}(f)) \quad (9)$$

As pointed above, the family of granulometries $\phi_n(f)$ is given by the morphological openings of f by a structuring function g. In the traditional works on gray-scale granulometry, g is chosen as a 3-D spherical or pyramidal structuring function, as depicted in the diagrams of Figure 1 below. The measurement function on f is made equal to the finite integral under f:

$$A(f) = \int_E f(x)dx \quad (10)$$

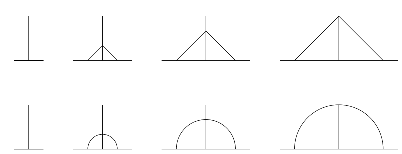


Figure 1: Cross sections of 3-D pyramidal and spherical structuring functions

4 Size-Intensity Granulometry

The gray-scale granulometry represented by the Size-Intensity Diagram is built by using a family of openings where the structuring function $g_{r,k}$ is a non-planar cylinder of radius r and amplitude k. Figure 2 shows the cross-sections of four elements of increasing radius and amplitude.

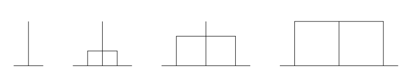


Figure 2: Cross sections of cylindrical structuring functions

In the first implementation of the SID ([Lot96], Section 4), we define the measurement function $m(f)$ as the number of the non-zero pixels of f. In this case each line of the diagram provides the binary granulometry of f thresholded at level k. Subsequent experiments with both real and synthetic image have shown, however, that the use of the volume as $m(f)$ provides more information regarding the granulometric curves. So we have changed the definition of $m(f)$ in accordance with (10):

$$A(f) = \int_E f(x)dx \tag{11}$$

We define the Size-Intensity Diagram as:

$$SID(f)(r,k) = A(\gamma_{g_{r,k}}(f)) \tag{12}$$

One can prove that the above granulometry satisfies the Matheron Axioms extended to two parameters:

$$\begin{gathered} \forall r \geq 0, \forall k \geq 0, \\ \gamma_{g_{r,k}} \text{ is increasing} \\ \forall r \geq 0, \forall k \geq 0, \\ \gamma_{g_{r,k}} \text{ is anti-extensive} \\ \forall r_1 \geq 0, \forall r_2 \geq 0, \forall k_1 \geq 0, \forall k_2 \geq 0, \\ \gamma_{g_{r1,k1}} \gamma_{g_{r2,k2}} = \gamma_{g_{max(r1,r2),max(k1,k2)}} \end{gathered} \tag{13}$$

Equation (12) thus represents a 2-D parameter space of axis r and k where the value at each coordinate (r, k) represents the volume of the opening of f by a primitive cylinder of radius r and amplitude k. By convention we use the r axis running horizontally from left to right and the k axis running vertically from top to bottom as shown in Figure 3.

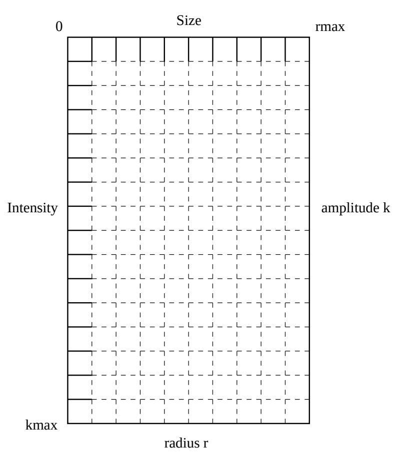


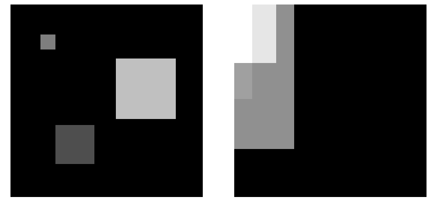
Figure 3: Size-Intensity Diagram

The column $SID(0, k)(f)$ contains the results of the opening by an structuring function containing only the origin with amplitude k. This column gives the number of pixels of f greater than k, so if we derive with respect to k and divide each point by the volume of the corresponding structuring function the information turns into the classical Intensity Histogram.

The row $SID(r, 0)(f)$ contains the results of the opening by a flat structuring element with increasing radius r. The same process of derivation and division by the volume pointed above can be used to obtain the classical gray-scale Pattern Spectrum [Vin94].

To be consistent with the Pattern Spectrum definition, we should take the gradient of SID to derive the corresponding Size-Intensity Histogram. Due to the inherent difficulty of the visualization of the gradient (magnitude and direction) in the bidimensional space, we propose the use of SID directly.

We implemented the Size-Intensity Diagram as an extension of MMach, a Mathematical Morphology Toolbox for the KHOROS System [Bar94]. Consequently fast algorithms are needed, which cannot be achieved by direct implementation of (12), since the opening operator become computationally prohibitive as the radius r of the structuring function increases. A complete discussion on the implementation of the Size-Intensity Diagram can be found on ([Lot96], Section 5).



(a) *(b)*

Figure 4: A synthetic gray-scale image (a) and its Size-Intensity Diagram (b)

5 Practical results and interpretations

We present in Figure 4 the Size-Intensity Diagram obtained from a synthetic image containing three squares of different sizes and amplitudes: a 5×5 square with amplitude 126, a 13×13 square with amplitude 78 and a 20×20, with amplitude 192. The granulometries were obtained using the 3×3 square as the size of the primitive structuring function.

Observing its Size-Intensity Diagram we notice that there are three steps in the radius direction (horizontally) corresponding to the three different radius of squares in the image. Towards the increasing amplitude direction (vertically) we also note steps corresponding to the three different amplitudes.

For illustrative purposes, we have also shown seven different texture images with the resulting SIDs. where we can see that the Size-Intensity Diagram gives different patterns for each texture.

The execution time to calculate each granulometry shown in Figure 5 was typically 3 minutes in a 486-DX2 66 MHz running the Linux operating system.

The two images in Figure 6 can be used to show how a Size-Intensity Diagram provides more information on image characterization than the classical gray-scale granulometry. For this we use a 256-level, 128×128 image with several small squares of different amplitudes in comparison with the same image, thresholded at an intermediate level selected in a way that the volume of both images are the same. While the SID clearly shows the results of thresholding, the Pattern Spectrum remains the same. It is caused by the fact that the intensity distribution information is lost when we divide each point by the volume of the structuring function.

As pointed above, the classical Intensity Histogram can be also obtained from the Size-Intensity Diagram by extracting its first column, $SID(0, k)$. Each point k of this column provides the number of points in the original image f that remain after the image is opened by $g_{(0,k)}$, which gives the points that are *greater* than k.

We can obtain the Intensity Histogram deriving the result with respect to k and dividing each point by the volume of the structuring function. Actually, the information provided by this column differs from the Intensity Histogram in the fact that the count for the points with value 0 is not made, since these points are regarded by the opening as having no information. Figure 7 shows the derived histogram for the three squares image of Figure 4.

6 Conclusions

In this paper we changed some definitions involved in the construction of the Size-Intensity Diagram - SID. The SID is a proposal of a gray-scale granulometry based on the family of openings with primitives cylinders of varying radius and amplitude and measuring the volume of the umbra of each opening. This form of granulometry can be reduced to the particular cases of the classical Intensity Histogram and to the classical binary granulometry. We recall that the classical gray-scale granulometry reported in the literature normally employs flat or spherical structuring functions and accounts only for the size of the image elements.

The Size-Intensity Diagram was implemented as a new operator of MMach, a Mathematical Mor-

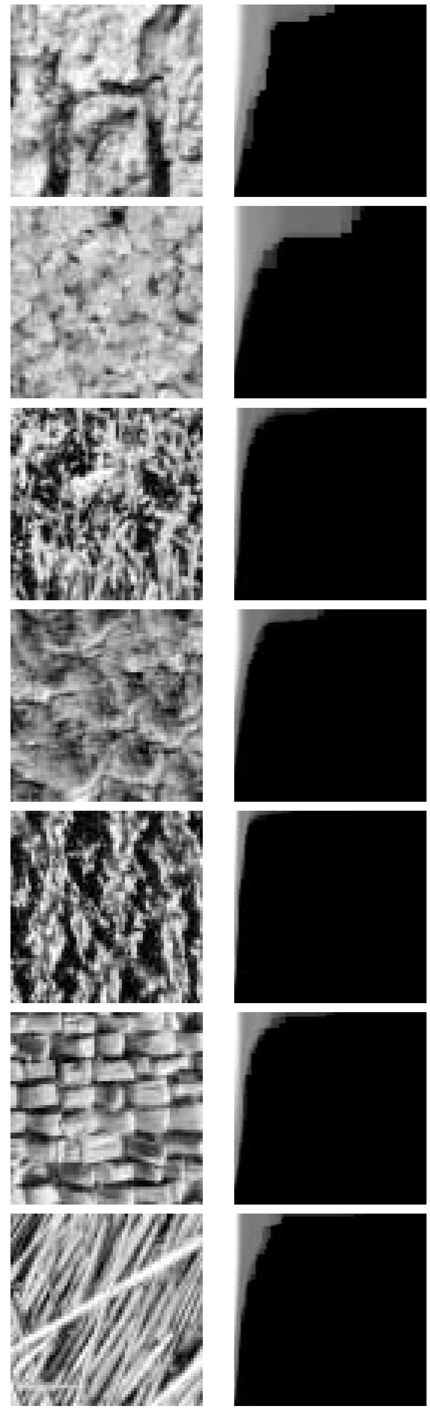


Figure 5: Texture images and their Size-Intensity Diagram pairs. 256-level images of size 64×64. The primitive structuring function used was the 3×3 square.

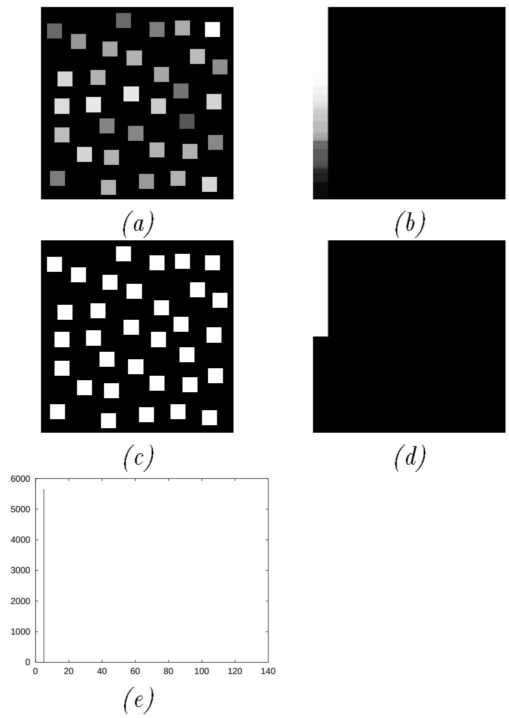


Figure 6: Comparison between the Size-Intensity Diagram and the Pattern Spectrum: (a) synthetic image, (c) image thresholded, (b,d) respective SID's, (e) pattern spectrum

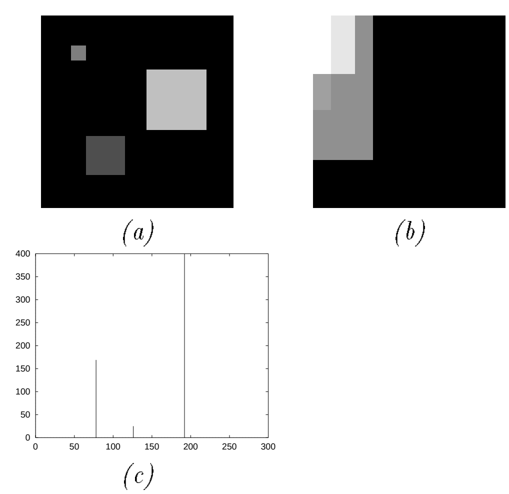


Figure 7: Comparison between Size-Intensity Diagram and the classical Intensity Histogram: (a) synthetic image, (b) SID and (c) Intensity Histogram

phology Toolbox for the KHOROS Image Processing and Visualization System. Although considerations about faster algorithms were made in ([Lot96], Section 4, the current implementation is still too computationally intensive for large images.

We have also shown an illustrative example which clearly indicates the superiority of the SID compared with the classical gray-scale granulometry.

The planned future development on this work are mainly four: make use of the Size-Intensity Diagram in real applications; further investigate better ways of visualizing the proposed granulometry (an interesting alternative seen to be the use of pseudo colors, by means of the choice of a colormap that preserves the relations of intensity), find a way of characterizing the SID of an image based on statistical parameters; and finally, search for a more efficient algorithm implementation.

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