CHARACTERIZATION OF LINEAR AND MORPHOLOGICAL OPERATORS

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INTRODUCTION

(1/1)

Linear and morphological operators are important pieces in image and signal processing.

Linear operators are models for many optical sensors and they can be used as filters for **sensor simulation** and **image restoration**.

Combinations of morphological operators can be used as filters for **image segmentation**.

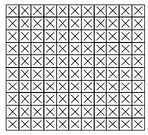
Operator characterization shows where are the differences and the similarities between the linear and morphological operators.

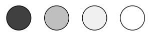
Operator characterization helps to understand how these operators are built.

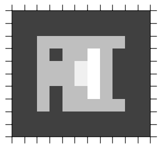
IMAGE PROCESSING OBJECTS

(1/4)

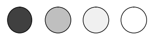
$$f: \ E \rightarrow K \quad \text{(image)}$$

$E \triangleq$ 

$K \triangleq \{$  $\}$

$f: E \rightarrow K \ \triangleq$ 

$$l: \ K_1 \rightarrow K_2 \quad \text{(gray–scale transform or lut)}$$

$K \triangleq \{$  $\}$

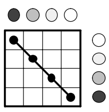
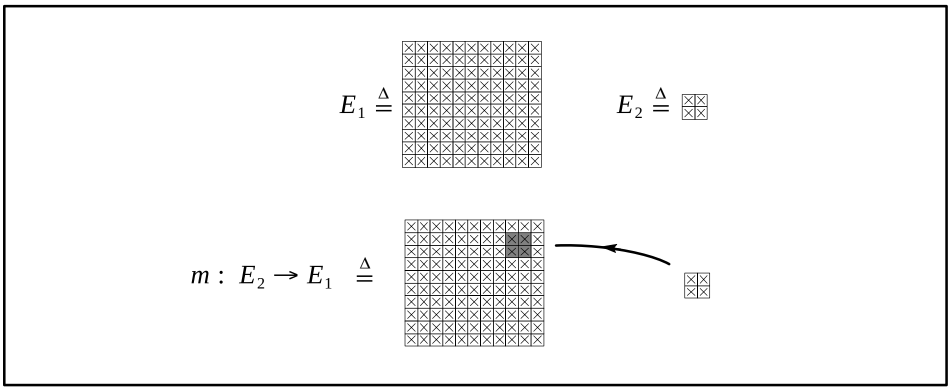
$l: K \rightarrow K \ \triangleq$ K  K

IMAGE PROCESSING OBJECTS

(2/4)

$m: \ E_2 \rightarrow E_1$ (domain transform)



$\mu: \ K_1^{\ E} \rightarrow K_2$ (measure)

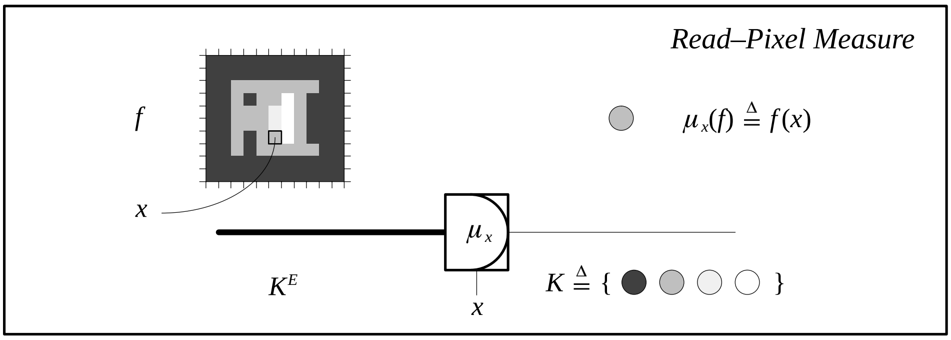
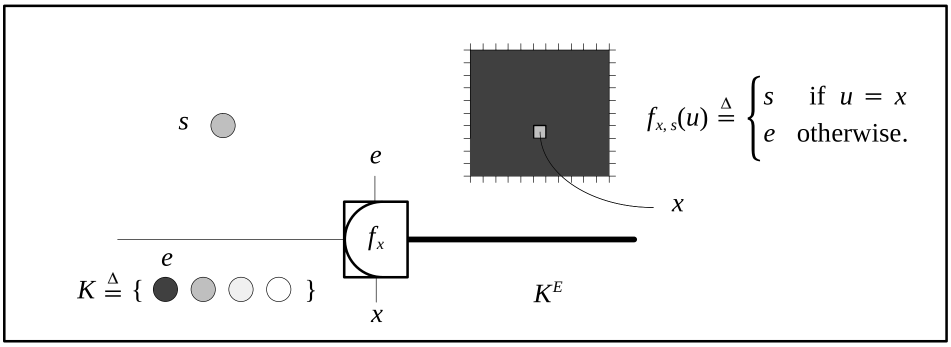


IMAGE PROCESSING OBJECTS

(3/4)

$f_x : \; K \to K^E$ (impulse image creator)



$\Psi : \; K_1^{E_1} \to K_2^{E_2}$ (operator)

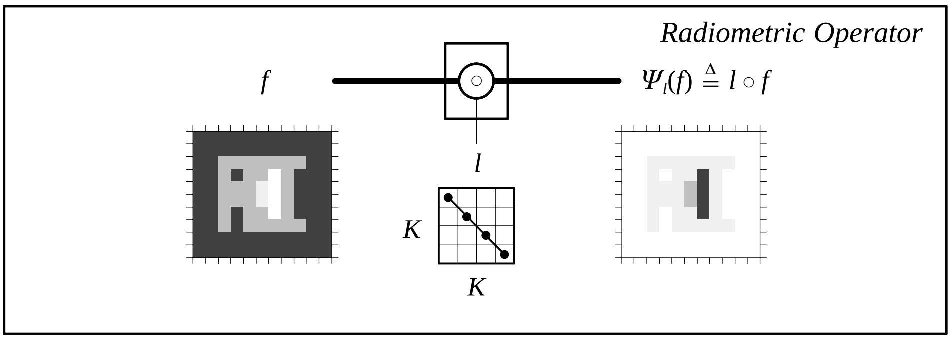
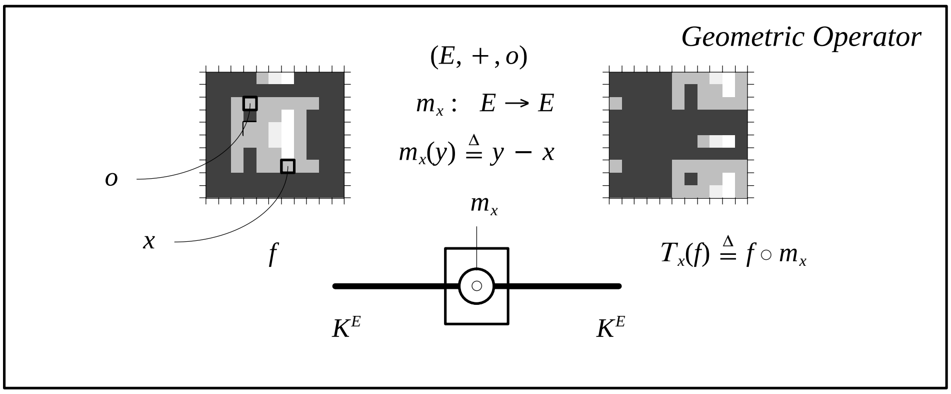


IMAGE PROCESSING OBJECTS

(4/4)



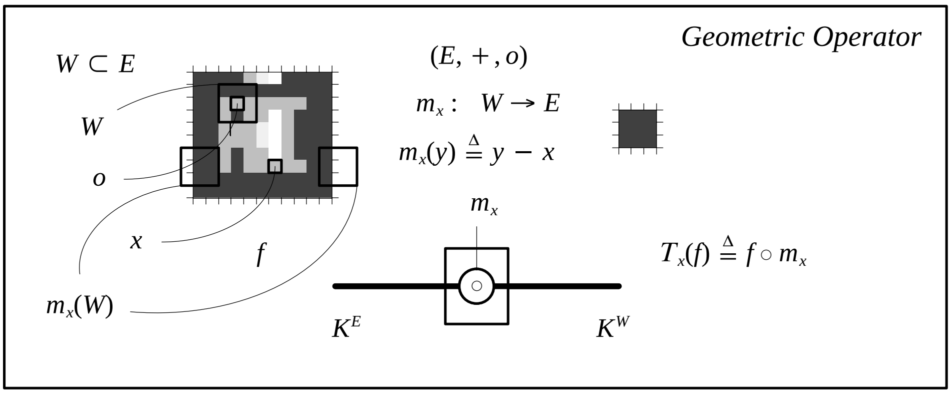


IMAGE OPERATIONS

(1/3)

Operations on gray–scale

$(K, \square, e)$ is a *commutative monoid* iff

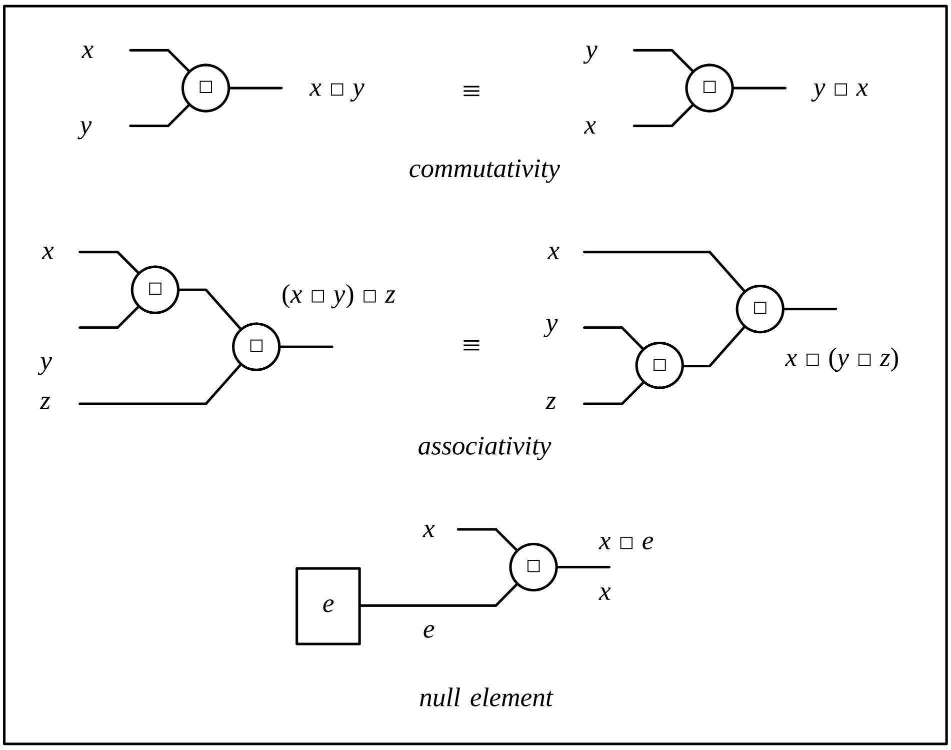


IMAGE OPERATIONS

(2/3)

Let $K \triangleq \mathbf{R}$ (set of real numbers) then
$(K, +, \cdot, 0)$ is a **linear vector space** and
$(K, +, 0)$ is a **commutative monoid**.

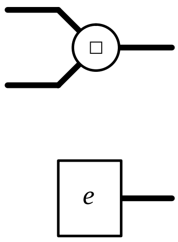
Let $K \triangleq [0, k] \subset \mathbf{Z}$ and let
$s \vee t \triangleq \max\{s, t\}$ and $s \wedge t \triangleq \min\{s, t\}$, then
$(K, \vee, \wedge)$ is a **lattice** and
$(K, \vee, 0)$ and $(K, \wedge, k)$ are two **commutative monoids**.

IMAGE OPERATIONS

(3/3)

Operation extension to images

$$f = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \qquad g = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$$



$$f \,\square\, g \triangleq \begin{bmatrix} (1 \,\square\, 5) & (5 \,\square\, 6) \\ (2 \,\square\, 2) & (4 \,\square\, 3) \end{bmatrix}$$

$$e \triangleq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$(K^E, \square, e)$ is a **commutative monoid**.

Let $K \triangleq \mathbf{R}$, then
$(K^E, +, o)$ is a **commutative monoid**.

Let $K \triangleq [0, k] \subset \mathbf{Z}$, then
$(K^E, \vee, o)$ and $(K^E, \wedge, i)$ are **commutative monoids**.

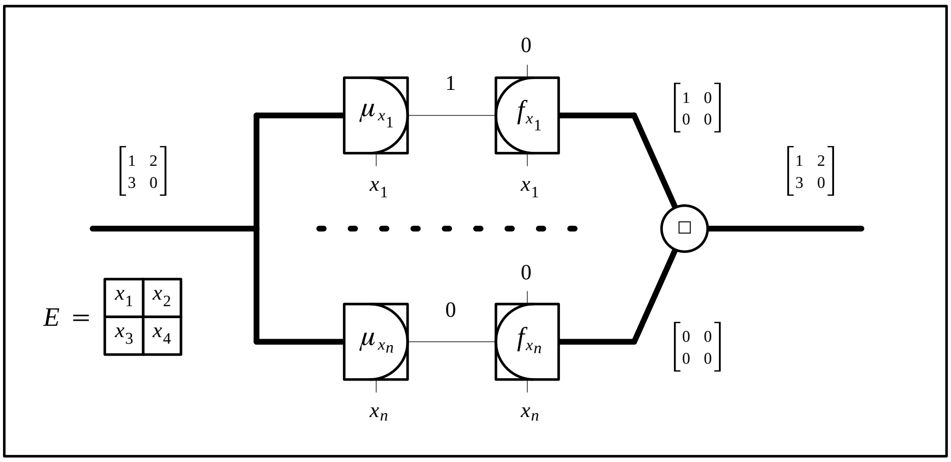
IMAGE DECOMPOSITION

(1/1)

Image decomposition
(consequence of $(K, \square, e)$ being a commutative monoid)

Let $E = \{x_1, x_2, ..., x_n\}$ and let $K = [0, k]$.
Example: $n = 4$ and $k = 4$.

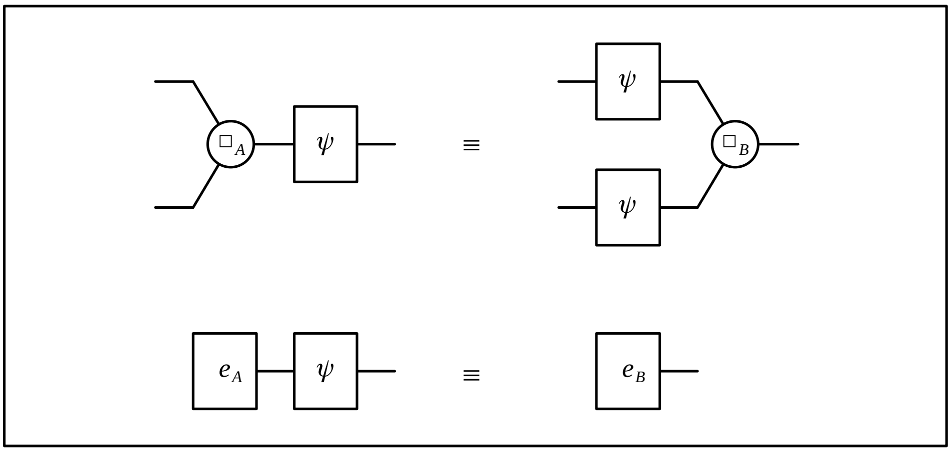
$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \square \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \square \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \square \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



MORPHISM OF COMMUTATIVE MONOIDS

(1/1)

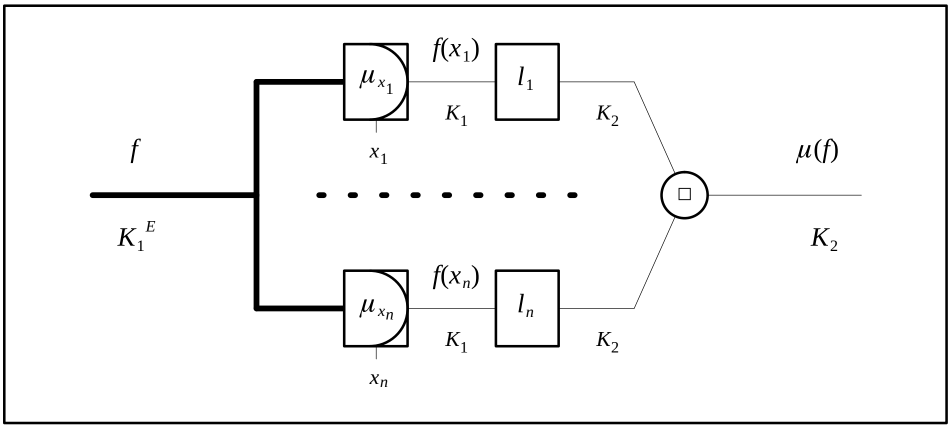
Let $(A, \square_A, e_A)$ and $(B, \square_B, e_B)$ be two commutative monoids.
A mapping ψ from A to B is a *morphism* iff



SEPARABLE MEASURES

(1/1)

Let $E = \{x_1, x_2, ..., x_n\}$.
A measure μ from $K_1{}^E$ to K_2 is *separable*
if there exists a family $\{l_1, l_2, ..., l_n\}$ of luts from K_1 to K_2
such that



Proposition (separability $\times$ morphism)
The measures which are morphisms of commutative monoids are separable and their characteristic luts are morphisms too.
(This is a consequence of the image decomposition.)

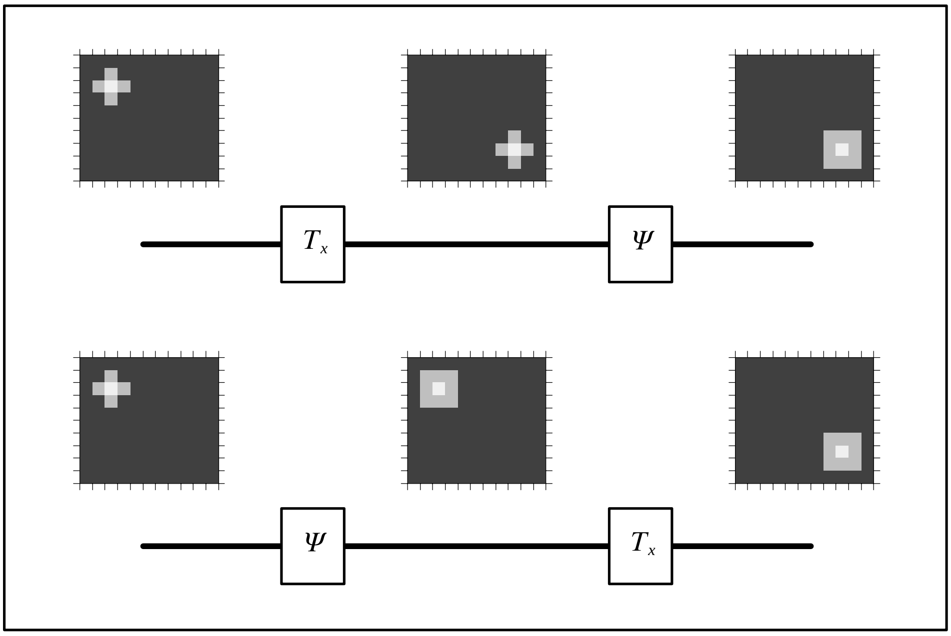
TRANSLATION INVARIANT OPERATORS

(1/1)

Let $(E, +, o)$ be an Abelian group.

Ψ is *translation invariant* (*ti–operator*) iff, for any x in E

$$T_x \circ \Psi = \Psi \circ T_x$$

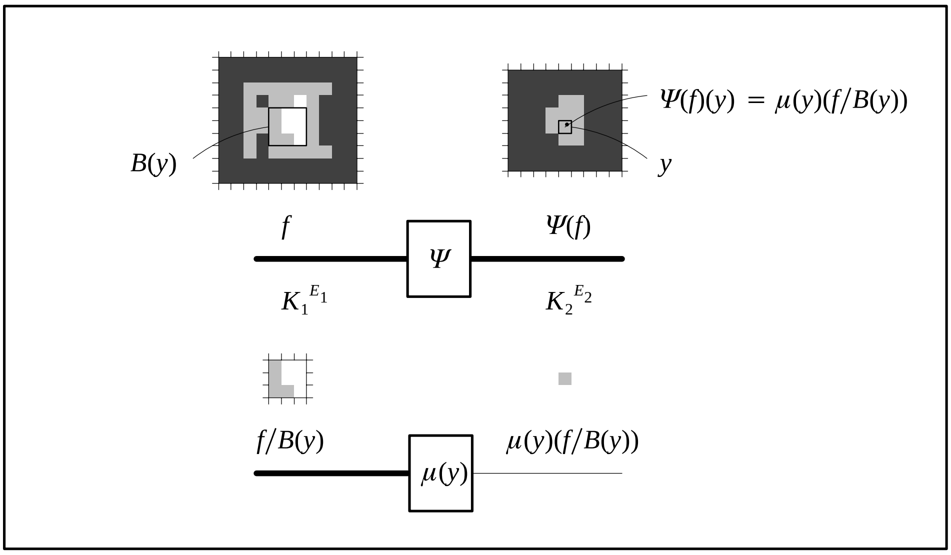


WINDOW OPERATORS

(1/1)

Let B be a mapping from E_2 to $\mathcal{P}(E_1)$.
Ψ is a *window operator* (*w–operator*) iff, for any y in E_2,
$f/B(y) = g/B(y) \Rightarrow \Psi(f)(y) = \Psi(g)(y)$.

Let Ψ be w–operator,
then there exists a family $\mu(\cdot)$ of measures such that

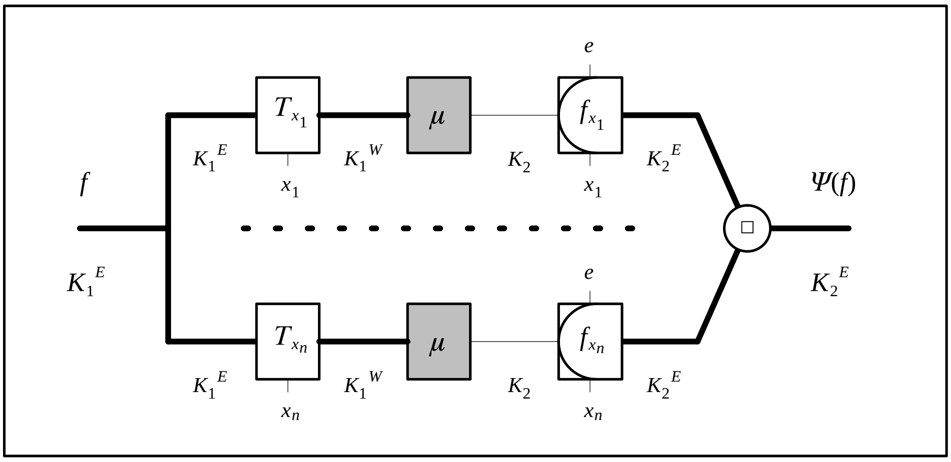


tiw–OPERATOR CHARACTERIZATION

(1/1)

Let $E = \{x_1, x_2, ..., x_n\}$.

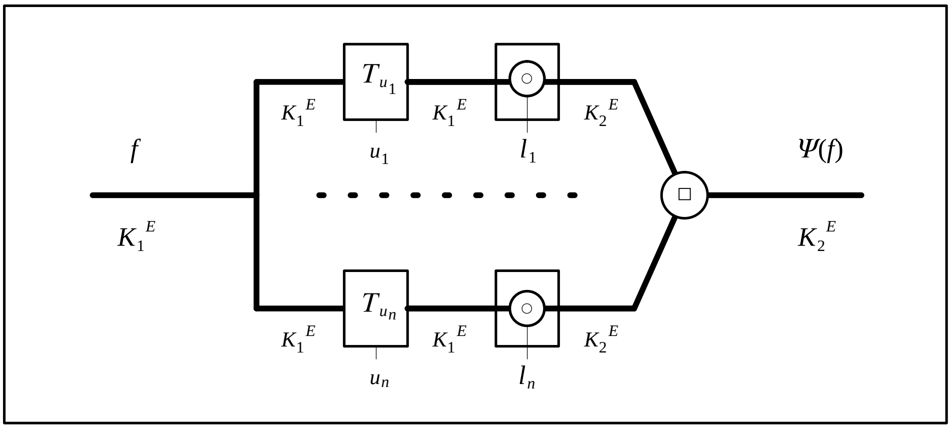
Let Ψ be tiw–operator, then there exists a measure μ such that.



tiw–MORPHISM CHARACTERIZATION

(1/1)

Let $W = \{u_1, u_2, ..., u_n\}$.
Let Ψ be tiw–morphism of commutative monoids,
then there exists a family $\{l_1, l_2, ..., l_n\}$ of luts
which are morphisms of commutative monoids
such that

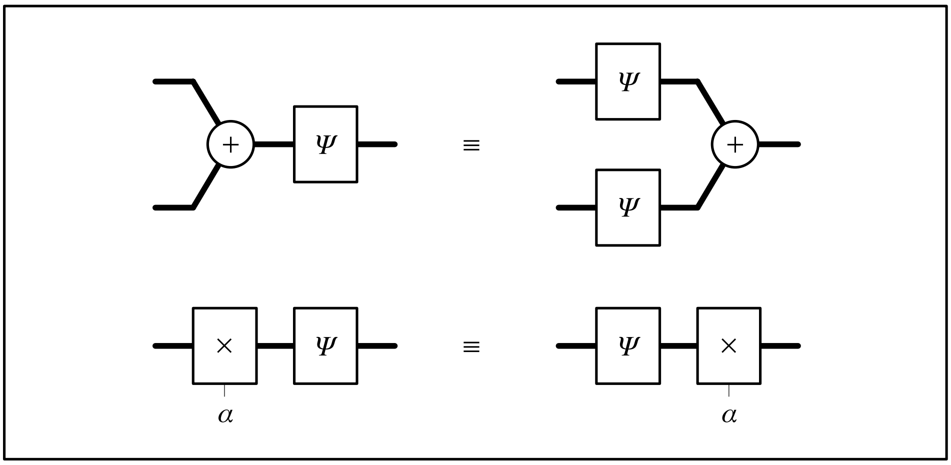


LINEAR OPERATOR CHARACTERIZATION
(1/3)

Let $K \triangleq \mathbf{R}$, then
$(K^E, +, \cdot, o)$ is a **linear vector space**.

A *linear operator* is a morphism
from $(K^E, +, \cdot, o)$ to $(K^E, +, \cdot, o)$.
In other words, for any f, g in K^E and α in K,

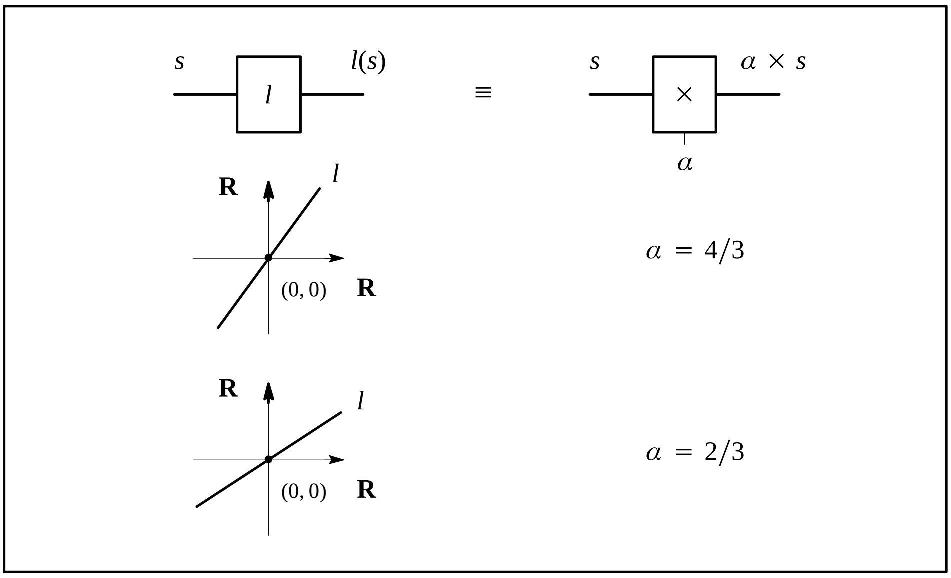
$$\boxed{\Psi(f + g) = \Psi(f) + \Psi(g) \quad \text{and} \quad \Psi(\alpha \cdot f) = \alpha \cdot \Psi(f).}$$



LINEAR OPERATOR CHARACTERIZATION

(2/3)

Let l be linear lut,
then there exists a real number α
such that

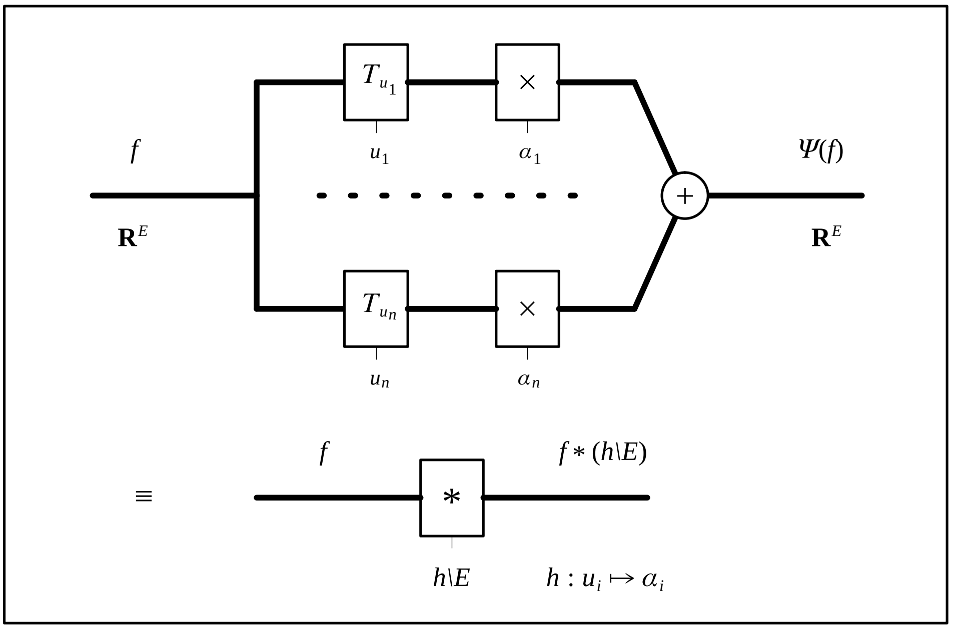


LINEAR OPERATOR CHARACTERIZATION

(3/3)

Let $W = \{u_1, u_2, ..., u_n\}$.

Let Ψ be linear tiw–operator on $\mathbf{R}^E$,
then there exists a family $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of real numbers such that



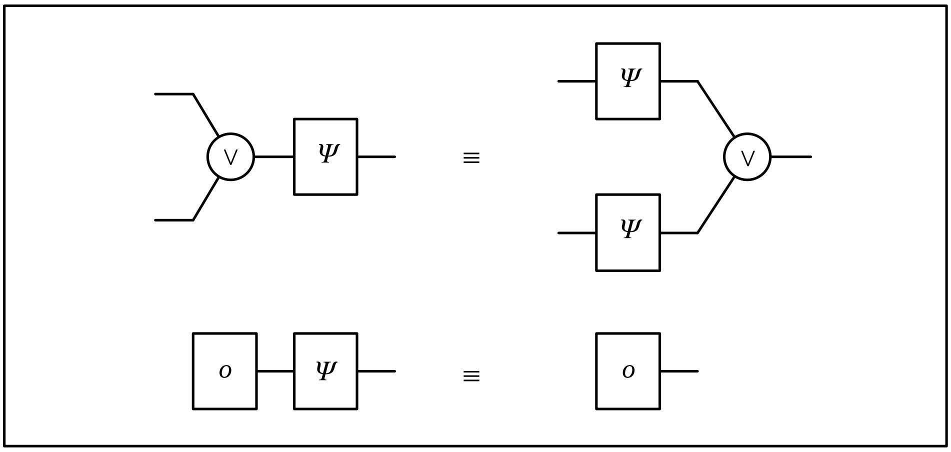
MORPHOLOGICAL OPERATOR CHARACTERIZATION

(1/7)

Let $K_1 \triangleq [0, k_1] \subset \mathbf{Z}$, $K_2 \triangleq [0, k_2] \subset \mathbf{Z}$ and
let $s \vee t \triangleq \max\{s, t\}$, then
$({K_1}^E, \vee, o)$ and $({K_2}^E, \vee, o)$ are two **commutative monoids**.

A *dilation* is a morphism
from $({K_1}^E, \vee, o)$ to $({K_2}^E, \vee, o)$.
In other words, for any f, g in ${K_1}^E$,

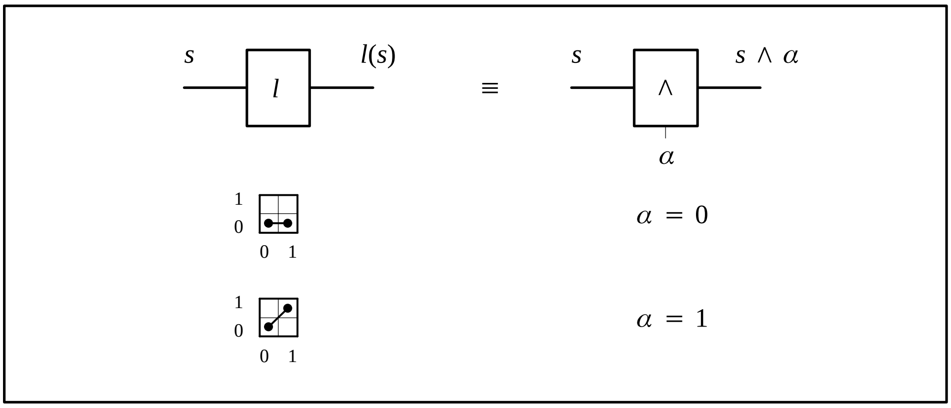
$$\boxed{\Psi(f \vee g) = \Psi(f) \vee \Psi(g) \quad \text{and} \quad \Psi(o) = o.}$$



MORPHOLOGICAL OPERATOR CHARACTERIZATION

(2/7)

Let l be a lut which is a dilation from $\{0, 1\}$ to $\{0, 1\}$, then there exists an integer number α in $\{0, 1\}$ such that

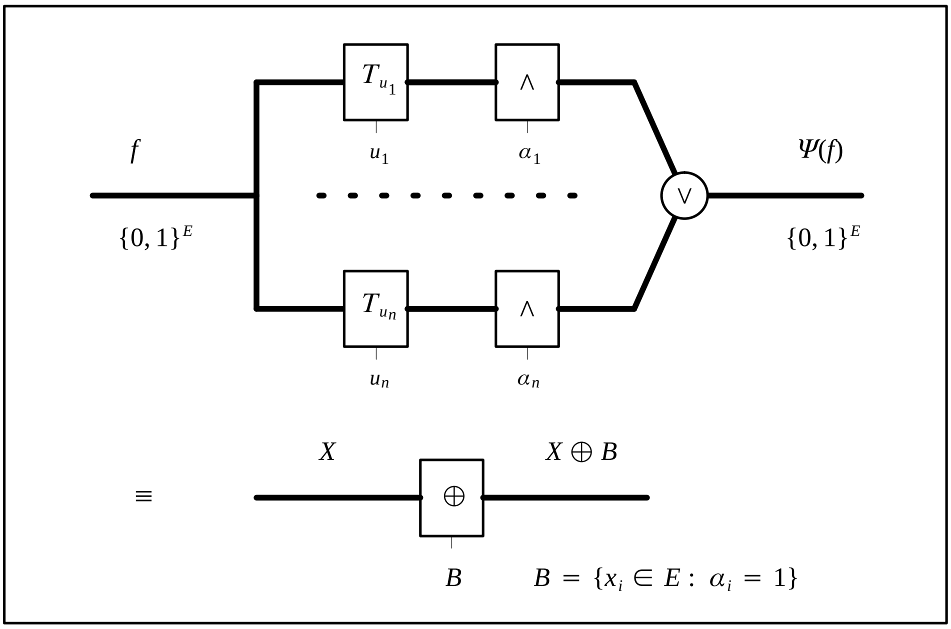


MORPHOLOGICAL OPERATOR CHARACTERIZATION

(3/7)

Let $W = \{u_1, u_2, ..., u_n\}$.

Let Ψ be tiw–operator which is a dilation on $\{0, 1\}^E$, then there exists a family $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of 0's and 1's such that

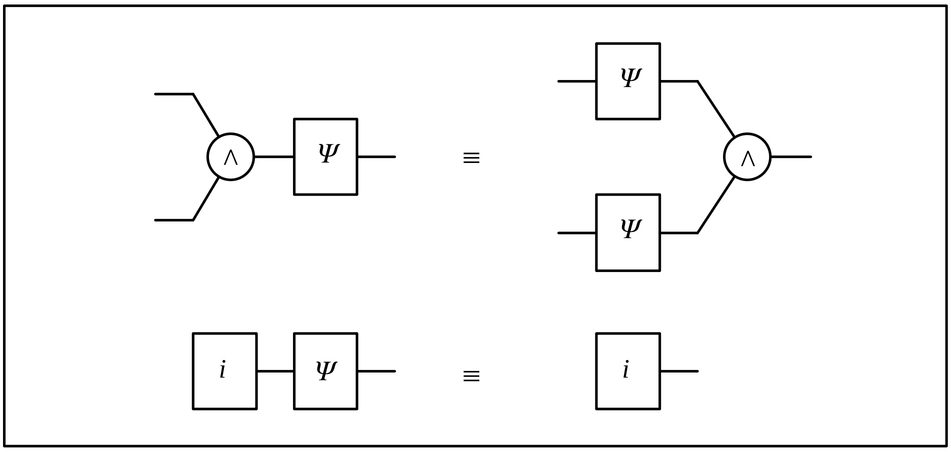


MORPHOLOGICAL OPERATOR CHARACTERIZATION
(4/7)

Let $K_1 \triangleq [0, k_1] \subset \mathbf{Z}$, $K_2 \triangleq [0, k_2] \subset \mathbf{Z}$ and
let $s \wedge t \triangleq \min\{s, t\}$, then
$(K_1{}^E, \wedge, i)$, and $(K_2{}^E, \wedge, i)$ are two **commutative monoids**.

An *erosion* is a morphism
from $(K_1{}^E, \wedge, i)$ to $(K_2{}^E, \wedge, i)$.
In other words, for any f, g in $K_1{}^E$,

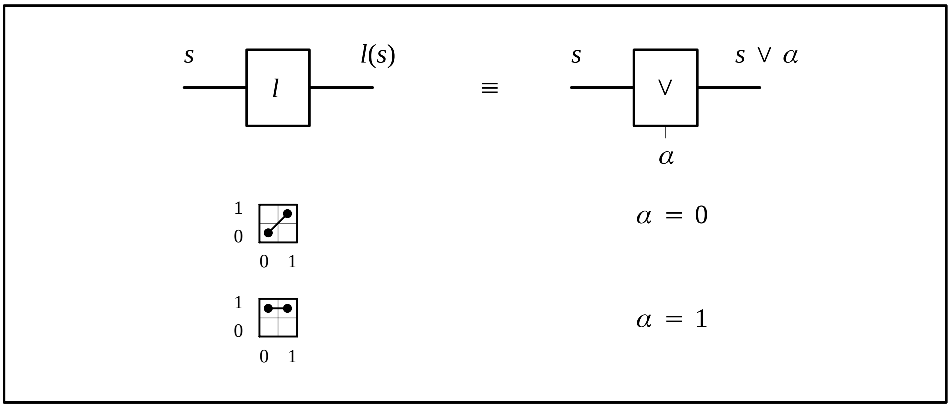
$$\boxed{\Psi(f \wedge g) = \Psi(f) \wedge \Psi(g) \quad \text{and} \quad \Psi(i) = i.}$$



MORPHOLOGICAL OPERATOR CHARACTERIZATION

(5/7)

Let l be a lut which is an erosion from $\{0,1\}$ to $\{0,1\}$,
then there exists an integer number α in $\{0,1\}$
such that

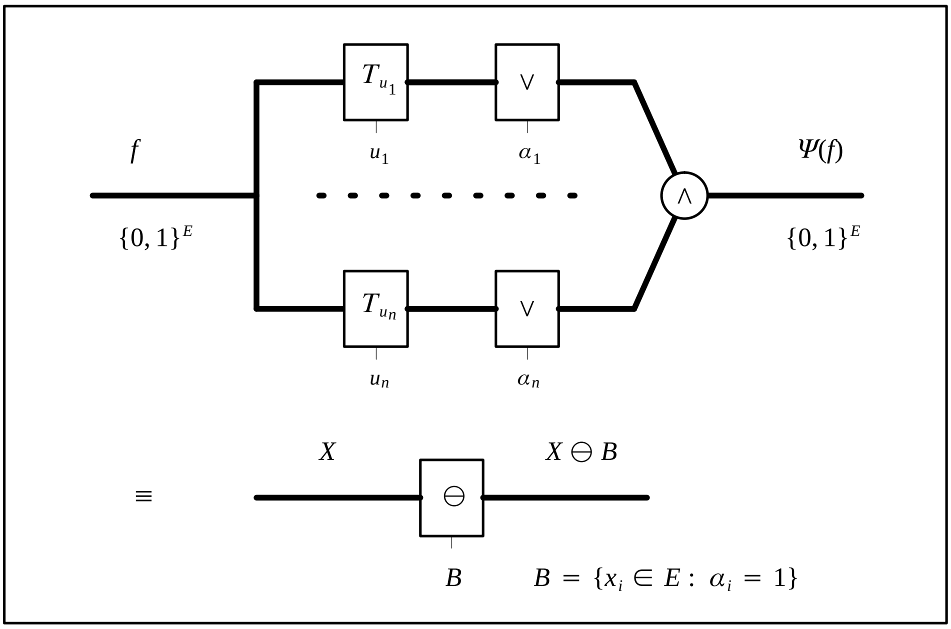


MORPHOLOGICAL OPERATOR CHARACTERIZATION

(6/7)

Let $W = \{u_1, u_2, ..., u_n\}$.

Let Ψ be tiw–operator which is an erosion on $\{0,1\}^E$, then there exists a family $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of 0's and 1's such that

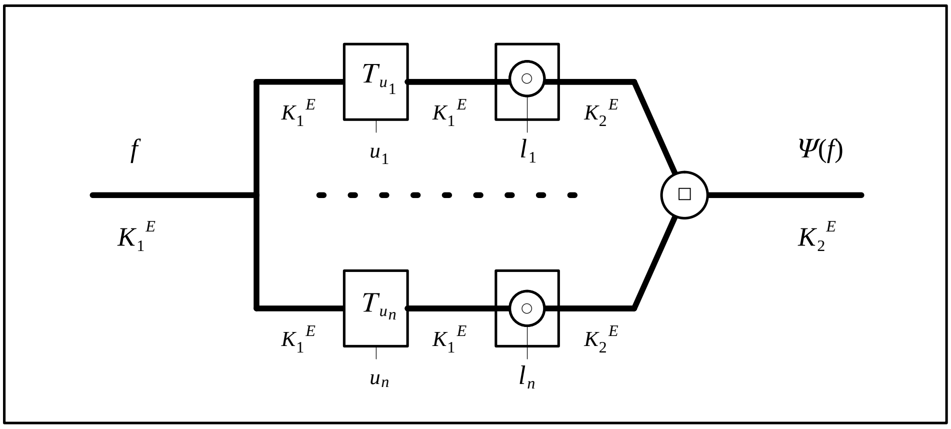


MORPHOLOGICAL OPERATOR CHARACTERIZATION

(7/7)

Non binary case

Let Ψ be tiw–operator which is a dilation (erosion) from $[0, k_1]^E$ to $[0, k_2]^E$, then there exists a family $\{l_1, l_2, ..., l_n\}$ of luts which are dilations (erosions) such that

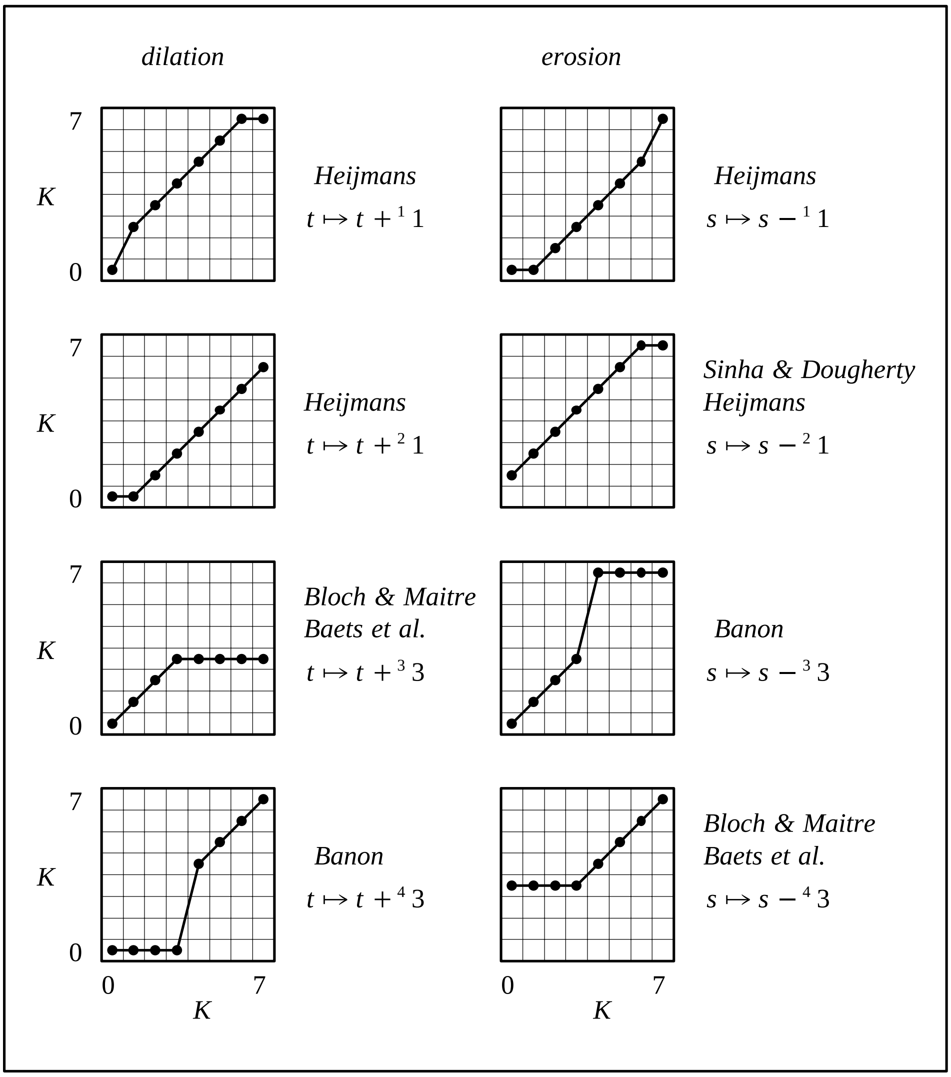


l is a dilation $\Leftrightarrow$ l is increasing and $l(0) = 0$

l is an erosion $\Leftrightarrow$ l is increasing and $l(k_1) = k_2$

Examples of morphological luts ($k_1 = k_2 = 7$)

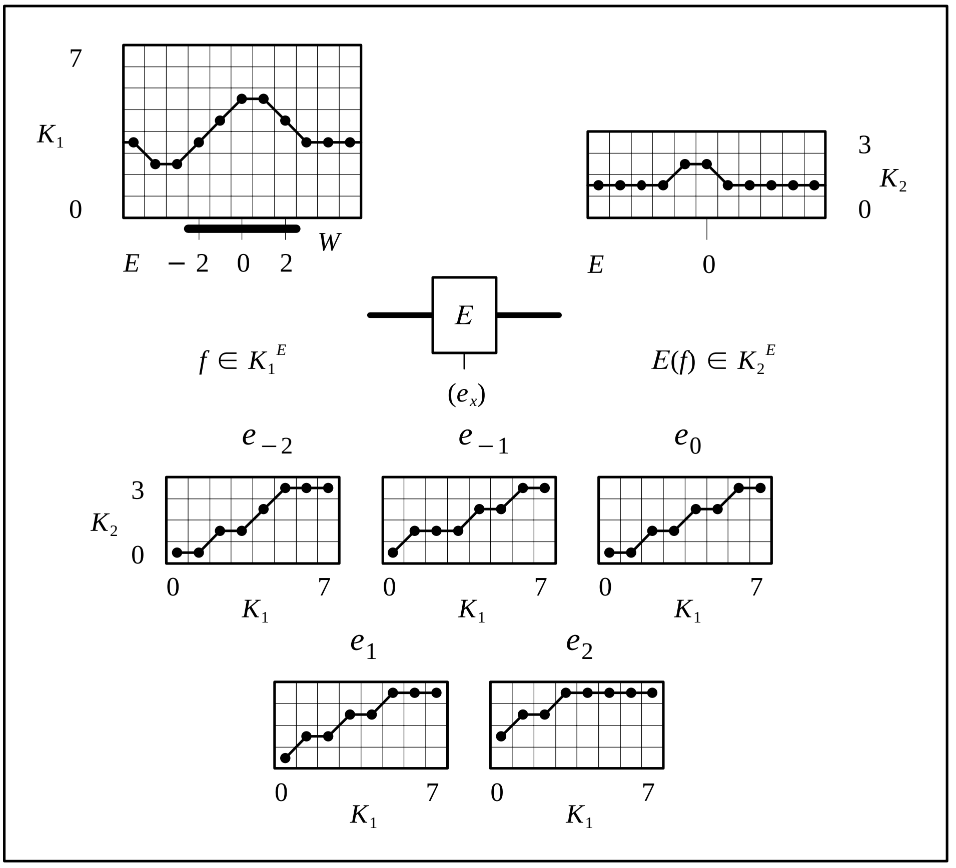
(see next transparency)


dilation
erosion
7
K
0
Heijmans
$t \mapsto t +^1 1$
Heijmans
$s \mapsto s -^1 1$
Heijmans
$t \mapsto t +^2 1$
Sinha & Dougherty
Heijmans
$s \mapsto s -^2 1$
Bloch & Maitre
Baets et al.
$t \mapsto t +^3 3$
Banon
$s \mapsto s -^3 3$
Banon
$t \mapsto t +^4 3$
Bloch & Maitre
Baets et al.
$s \mapsto s -^4 3$
0
7
K

EROSION EXAMPLE

(1/1)

Non binary case



REFERENCES

G. J. F. Banon, “Characterization of translation invariant elementary operators for gray–level morphology”, “Neural, Morphological, and Stochastic Methods in Image and Signal Processing”, Editors: E. Dougherty, F. Preteux and S. Shen, SPIE vol. 2568, 1995.
UR*Lib* repository: <dpi.inpe.br/banon/1997/09.22.18.06>

G. J. F. Banon, “Characterization of translation–invariant elementary morphological operatores between gray–level images”, Research Report INPE–5616–RPQ/671, Instituto Nacional de Pesquisas Espaciais, São José dos Campos, SP, Brazil, 1995.
UR*Lib* repository: <dpi.inpe.br/banon/1995/10.20.17.45>

G. J. F. Banon, “Formal introduction to digital image processing”, Booklet INPE–6969–PUD/034, Instituto Nacional de Pesquisas Espaciais, São José dos Campos, SP, Brazil, 1998. 133 p.
UR*Lib* repository: <dpi.inpe.br/banon/1998/07.02.12.54>

H. J. A. M. Heijmans, “Theoretical aspects of gray–level morphology”, IEEE trans. on PAMI, Vol. PAMI–13, No. 6, pp 568–582, June, 1991.

H. J. A. M. Heijmans, “Morphological Image Operators”, Advances in Eletronics and Electron Physics. Supplement 24. Academic Press, Inc., Boston, 1994.