# Projective estimators for point/tangent representations of planar curves

Thomas Lewiner and Marcos Craizer
Department of Mathematics, PUC–Rio de Janeiro, Brazil
http://www.mat.puc-rio.br/ {tomlew,craizer}

## Abstract

*Recognizing shapes in multiview imaging is still a challenging task, which usually relies on geometrical invariants estimations. However, very few geometric estimators that are projective invariant have been devised. This paper proposes projective length and projective curvature estimators for plane curves, when the curves are represented by points together with their tangent directions. In this context, the estimations can be performed with only the four point-tangent samples for the projective length and five for the projective curvature. The proposed length estimator is based on affine estimators and is proved to be convergent. The curvature estimator relies on the length to fit logarithmic spirals to the point-tangent samples. It is projective invariant and experiments indicate its convergence. Preliminary results using both estimators together are promising, although the estimators' lack of robustness would require additional work for noisy cases.*

**Keywords:** *Projective Differential Geometry, Projective Curvature, Projective Lenght, Discrete estimators.*

## 1. Introduction

Computer Vision applications usually deal with images that are two-dimensional projections of three-dimensional scenes. Different projections of the same scene can be identified by isolating and matching the scene elements in each projection. This matching usually relies on quantities that are invariant by the projective group [5, 9, 2]. Projective length and projective curvature are the two simplest such quantities in differential geometry. Together, they are sufficient to describe a planar curve up to a projective transformation ([7]). However, their estimation tends to be very sensitive to noise, since for a parametric curve, they depend on the fifth and seventh order derivatives respectively. This paper proposes numerically stable projective length and curvature estimators for planar curves.

Instead of considering discrete curves as a sequence of points, we choose here to sample a planar curve associating to each point its tangent direction. In Computer Vision, the curves of a scene are usually obtained by edge detection, which naturally generate these point-tangent samples. In this context, the projective length estimator uses only four point-tangent samples and the projective curvature estimator uses five of them. The same model was considered in [6] to define affine length and affine curvature estimators. These affine quantities are used here to estimate the derivative of the affine curvature which leads to our projective length estimator. The proposed projective length estimator is proved to be convergent and numerical experiments included in this work show its numerical stability. We then estimate the projective curvature through the frames estimates at three consecutive samples. For this task, we fit logarithmic spirals to the point-tangent samples. These spirals have projective curvature zero, similarly to polygonal lines in the Euclidean case [16].

The knowledge of the projective lengths allows adjusting such spirals with only three point-tangent samples. When the exact projective lengths are known a priori, the estimator proposed here is stable, and numerical experiments indicate its convergence. When one uses the length estimator to estimate the projective curvature, experimental results remains promising, but numerical problem also appears, especially when the projective lengths are small.

**Related works.** The study of analytic expression for projective curvature is laborious. However, Faugeras [7] describes very nicely the Euclidian, affine and projective geometry and evolutions of plane curves, with explicit formulas for projective length and curvature.

For affine quantities, the definition of affine invariants for discrete curves has been studied in several works. Callabi *et al.* [4, 3] propose affine length and curvature estimators with convergence proofs for curves given by a sequence of points, while Craizer *et al.* [6] define estimators for curves given by points and tangent directions. These estimators are particular combinations of *joint invariants*, which are functions of the points' coordinates that are invariant under a given group action. Boutin [1] proposed joint invariants for Euclidian and affine groups. Olver [13] describes how to construct joint invariants for any group. In particular, he described all joint invariants for the affine group in the plane.

The authors are not aware of any previous work that explicitly estimates projective lengths and curvatures for discrete curves. The probable reason for this absence is that these concepts deal with high order derivatives, which in general are numerically unstable. However, several works try to define projective quantities, in particular in multi-view images [11, 10, 14]. In particular, Lazebnik and Ponce [15] implement some notions of oriented projective geometry, introduced by Stolfi [12], to characterize silhouette features.

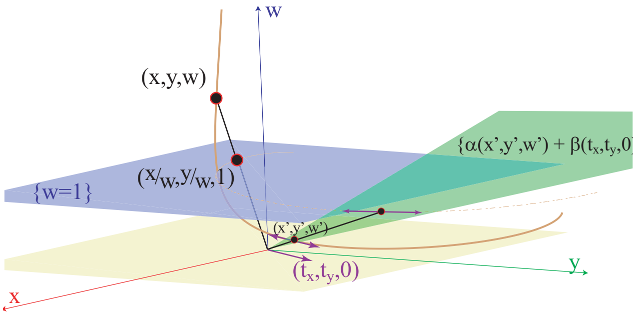


**Figure 1. Homogeneous coordinates: point $(\frac{x}{w}, \frac{y}{w}, 1)$ is the projection of point $(x, y, w)$ onto the plane $\{w = 1\}$. They are projective equivalent points. Similarly, any line in the plane $\{\alpha \cdot (x', y', w') + \beta \cdot (t_x, t_y, 0)\}$ is projective equivalent to the tangent line at $(x', y', w')$.**

# 2. Preliminaries

In this section, we shall review the definitions of the basic quantities associated with smooth planar curves that are invariant under the special affine group and under the projective group. We will denote a parametric curve $C$ in homogeneous coordinates as $\mathbf{x}(t) = (x(t), y(t), w(t))$. The same curve $C$ in planar coordinates will be denoted by $\mathbf{x}(t) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)})$. We will use the planar notation for affine quantities and the homogeneous one for projective quantities.

The determinant of three vectors is denoted $|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3|$, and the determinant of two planar vectors will be denoted $|\mathbf{x}_1, \mathbf{x}_2|$. With this notation, a strictly convex curve $C$ satisfies $|\mathbf{x}'(t), \mathbf{x}''(t)| \neq 0$ in planar coordinates, and $|\mathbf{x}''(t), \mathbf{x}'(t), \mathbf{x}(t)| \neq 0$ in homogeneous coordinates.

## 2.1. Affine invariant geometry

In this subsection, we describe the simplest quantities associated with a convex plane curve that are invariant under the special affine group, i.e., the group of linear transformations with determinant 1. There are two basic quantities: the affine length and the affine curvature. Any convex plane curve can be recovered, up to an affine transformation, from these quantities.

Assume that the convex curve $C$ satisfies $|\mathbf{x}'(t), \mathbf{x}''(t)| > 0$. The *affine length* $s$ is then defined by

$$s(t) = \int_0^t \sqrt[3]{|\mathbf{x}'(u), \mathbf{x}''(u)|}du.$$

If one takes $s$ as a new parameter then $|\mathbf{x}'(s), \mathbf{x}''(s)| = 1$. This implies that $|\mathbf{x}'(s), \mathbf{x}'''(s)| = 0$ and that one can write $\mathbf{x}'''(s) = -\mu(s)\mathbf{x}'(s)$. The scalar $\mu(s)$ is called the *affine curvature* of $C$ at the point $\mathbf{x}(s)$.

Solving the differential equation $\mathbf{x}'''(s) = 0$, one obtains the plane curves of zero affine curvature. The set of solutions of this equation are parabolas that can be described in homogeneous coordinates as $\mathbf{x}(s) = T_{aff} \cdot \mathbf{A}(s)$, where

$$T_{aff} = \begin{bmatrix} A & B & C \\ D & E & F \\ 0 & 0 & 1 \end{bmatrix} \quad and \quad \mathbf{A}(s) = \begin{bmatrix} s \\ \frac{s^2}{2} \\ 1 \end{bmatrix}. \quad (1)$$

## 2.2. Projective invariant geometry

In this subsection, we describe the corresponding quantities for the projective group. In homogeneous coordinates, a projective transformation is defined by an invertible linear transformation of $\mathbb{R}^3$. Since, from the projective point of view, two points in $\mathbb{R}^3$ that are in the same line through the origin are equivalent (see Figure 1), linear transformations of the form $T$ and $\lambda T$ are projective equivalent, for any $\lambda \neq 0$. We will thus consider a projective transformation as a $3 \times 3$ matrix of determinant 1.

**Projective length.** Assuming that $|\mathbf{x}'', \mathbf{x}', \mathbf{x}| > 0$, we can decompose $\mathbf{x}'''$ on the frame $(\mathbf{x}, \mathbf{x}', \mathbf{x}'')$, obtaining: $\mathbf{x}''' + p\mathbf{x}'' + q\mathbf{x}' + r\mathbf{x} = 0$, where

$$p = -\frac{|\mathbf{x}''', \mathbf{x}', \mathbf{x}|}{|\mathbf{x}'', \mathbf{x}', \mathbf{x}|}, \quad q = \frac{|\mathbf{x}''', \mathbf{x}'', \mathbf{x}|}{|\mathbf{x}'', \mathbf{x}', \mathbf{x}|}, \quad r = -\frac{|\mathbf{x}''', \mathbf{x}'', \mathbf{x}'|}{|\mathbf{x}'', \mathbf{x}', \mathbf{x}|}.$$

Consider the function $H = r - \frac{1}{3}pq + \frac{2}{27}p^3 - \frac{1}{2}q' + \frac{1}{3}pp' + \frac{1}{6}p''$. Assuming that $H(t) \neq 0$, one defines the *projective length* $\sigma$ by

$$\sigma(t) = \int_0^t \sqrt[3]{H(u)}du.$$

If one takes $\sigma$ as a new parameter for the curve, then $H(\sigma) = 1$. The function $H(t)$ has the following alternative definition: Suppose that the curve is parameterized by affine arc length $s$, then, following [7],

$$H(s) = \frac{1}{2}\mu'(s). \tag{2}$$

So the condition $H(t) \neq 0$ can be rewritten as $\mu'(s) \neq 0$.

**Projective curvature.** Since $\mathbf{x}(\sigma)$ and $\lambda(\sigma)\mathbf{x}(\sigma)$ are equivalent curves, we can force $p(\sigma)$ to be zero by choosing $\lambda(\sigma) = \exp\left(\frac{1}{3}\int_0^\sigma p(\tau)d\tau\right)$ [7]. Thus,

$$\mathbf{x}'''(\sigma) + q(\sigma)\mathbf{x}'(\sigma) + r(\sigma)\mathbf{x}(\sigma) = 0. \tag{3}$$

Since $1 = H(\sigma) = r(\sigma) - \frac{1}{2}q'(\sigma)$, one can write $q(\sigma) = 2k(\sigma)$ and $r(\sigma) = k'(\sigma) + 1$. The number $k(\sigma)$ is called *projective curvature*.

**Curves of zero projective curvature.** In the normalized form (3), a zero projective curvature curve $\mathbf{x}(\sigma)$ satisfies the differential equation $\mathbf{x}'''(\sigma) + \mathbf{x}(\sigma) = 0$. A particular solution of this differential equation is the logarithmic spiral: $\mathbf{P}(\sigma) = (P_x(\sigma), P_y(\sigma), P_w(\sigma))$, with

$$\begin{cases} P_x(\sigma) = \exp(\frac{1}{2}\sigma)\cos(\frac{\sqrt{3}}{2}\sigma) \\ P_y(\sigma) = \exp(\frac{1}{2}\sigma)\sin(\frac{\sqrt{3}}{2}\sigma) \\ P_w(\sigma) = \exp(-\sigma) \end{cases}$$

Any other solution is given by $T \cdot \mathbf{P}(\sigma)$, where $T$ is a projective transformation (see [8]). The set of logarithmic spirals of zero projective curvature is thus an 8-dimensional space.

## 3. A projective length estimator based on affine estimators

In this paper, we consider the discretisation of the curve $C$ as a sequence $\{(\mathbf{x}_i, \mathbf{x}'_i)\}_{1\leq i\leq n}$ of point-tangent samples. The vector $\mathbf{x}'_i$ only indicates the direction of the tangent to the curve, and its magnitude has no particular meaning. In this section, we describe the affine estimators of [6] for this model since we will use them for estimating the projective length. Moreover, they have a similar structure as the projective estimators. However, they will not be projective invariant, which may harm the projective curvature stability (see Figure 2).

### 3.1. Affine estimators

Denote by $s_i$ the affine length of the arc of $C$ between $(\mathbf{x}_i, \mathbf{x}'_i)$ and $(\mathbf{x}_{i+1}, \mathbf{x}'_{i+1})$, by $\mu_i$ the affine curvature at $\mathbf{x}_i$ and by $\nu_i$ the derivative of the affine curvature at $\mathbf{x}_i$.

**Affine length estimator.** For any pair of samples $(\mathbf{x}_i, \mathbf{x}'_i)$ and $(\mathbf{x}_{i+1}, \mathbf{x}'_{i+1})$, there exists a unique parabolic arc passing through $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$, being tangent to $\mathbf{x}'_i$ and $\mathbf{x}'_{i+1}$ at these points. The *affine length* $L_i$ of this parabolic arc is given by $2A_i^{1/3}$, where $A_i$ is the area of the triangle whose vertices are $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ and the intersection of the lines defined by $(\mathbf{x}_i, \mathbf{x}'_i)$ and $(\mathbf{x}_{i+1}, \mathbf{x}'_{i+1})$. The length $L_i$ is an estimator for $s_i$. An estimator for the affine length of $C$ is $\sum_{i=1}^{n-1} L_i$. In appendix A, it is shown that $L_i = s_i + O(s_i^5)$. Therefore, the estimator for the affine length of $C$ is convergent.

**Affine frame.** The above parabola can also be seen as an affine transformation of a basic arc of parabola $\mathbf{A}(s)$, $0 \leq s \leq L_i$, defined by formula (1). Let $T_{\mathit{aff}}$ be the affine transformation that transforms $(\mathbf{A}(0), \mathbf{A}'(0))$ and $(\mathbf{A}(L_i), \mathbf{A}'(L_i))$ to $(\mathbf{x}_i, \mathbf{x}'_i)$ and $(\mathbf{x}_{i+1}, \mathbf{x}'_{i+1})$. The *affine frame* at sample $i$ is then $(\mathbf{q}'_i, \mathbf{q}''_i) = T_{\mathit{aff}} \cdot (\mathbf{A}(0), \mathbf{A}'(0))$.

**Affine curvature estimator.** For three consecutive samples, consider the frames $(\mathbf{q}'_{i-1}, \mathbf{q}''_{i-1})$ and $(\mathbf{q}'_i, \mathbf{q}''_i)$ as above. An estimator of the third derivative is given by

$$\mathbf{q}'''_i = \frac{2(\mathbf{q}''_i - \mathbf{q}''_{i-1})}{L_{i-1} + L_i}.$$

Thus, an affine curvature estimator at sample $i$ is given by

$$\overline{\mu}_i = \frac{2|\mathbf{q}''_{i-1}, \mathbf{q}''_i|}{L_{i-1} + L_i}.$$

The estimator for $\int \mu ds$ is given by $\sum_{i=2}^{n-1} |\mathbf{q}''_{i-1}, \mathbf{q}''_i|$. In appendix A, it is shown that the affine curvature estimator satisfies $\overline{\mu}_i = \mu_i + O(s_{i-1} + s_i)$. Therefore, the estimator for the integral of the affine curvature is convergent.

**First derivative of the affine curvature.** For four consecutive samples, consider the affine curvatures $\overline{\mu}_i$ and $\overline{\mu}_{i+1}$. The estimator for the derivative of the affine curvature is given by

$$\overline{\nu}_i = \frac{10\,(\overline{\mu}_{i+1} - \overline{\mu}_i)}{3L_{i-1} + 4L_i + 3L_{i+1}},$$

In appendix A, it is shown that, under reasonable sampling conditions, $\overline{\nu}_i = \nu_i + O(s_{i-1} + s_i + s_{i+1})$.

Any plane curve can also be recovered, up to affine transformations, from its affine signature, which is the pair $(\mu(s), \mu'(s))$. This kind of representation is interesting for computer vision, since $\mu(s)$ and $\mu'(s)$ can be estimated locally, while the affine length is a global quantity. The pair $(\overline{\mu}_i, \overline{\nu}_i)$, $2 \leq i \leq n-2$, is an affine invariant signature estimator of the curve $C$ [1].

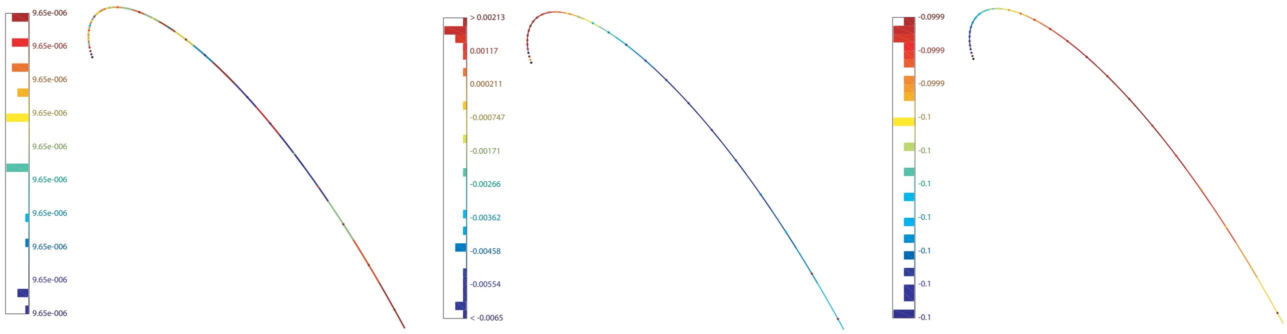

**Figure 2. The impact of projective length estimation error on the projective curvature estimator. Given a spiral of constant projective curvature with the noise of numeric operations, the projective estimator remains correct: the histogram is almost constant on the left image. However, with an approximate length estimated from affine quantities (right), the curvature estimator becomes unstable (middle).**

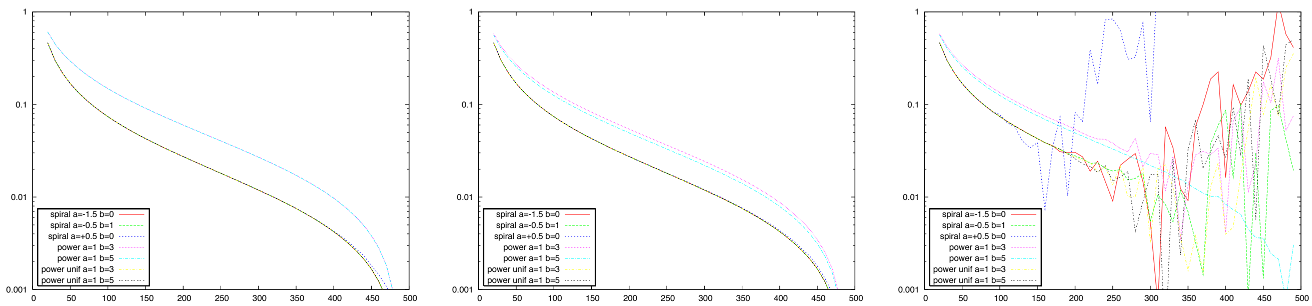


**Figure 3. Convergence test: relative error of the estimators (in logarithmic scale) vs. the number of samples on constant projective curvature curves. The curves chosen were $(e^{a \cdot t} \cos((b + \sqrt{3}/2)t), e^{a \cdot t} \sin((b+\sqrt{3}/2)t), e^{-t})$, uniformly sampled, and $(t^a, t^b, 1)$, uniformly sampled and also with a different sampling, uniformly in $t$. We have estimated the relative error $|\overline{\sigma} - \sigma|/\sigma$ of the length integral (left), the relative error $|\overline{k} - k|/k$ of the curvature integral using the exact analytic length for the spiral fitting (middle), and the same relative error of the curvature integral, but using the estimated length for the spiral fitting (right).**

### 3.2. Projective length estimator

For four consecutive samples, one can estimate the projective length of the arc of $C$ between $(\mathbf{x}_i, \mathbf{x}'_i)$ and $(\mathbf{x}_{i+1}, \mathbf{x}'_{i+1})$ by formula (2). Thus, the projective length can be estimated by

$$\overline{\sigma}_i = \frac{3L_{i-1} + 4L_i + 3L_{i+1}}{10} \cdot \sqrt[3]{\frac{\overline{\nu}_i}{2}}.$$

An estimator for the projective length of $C$ is thus given by $\overline{\sigma} = \sum_{i=2}^{n-2} \overline{\sigma}_i$. Assuming that $\mu'(s) \neq 0$, one can show that $\overline{\sigma}_i = \sigma_i + O(\sigma_i^2)$. Hence, the estimator $\overline{\sigma}$ for the projective length of $C$ is convergent. Although convergent, this projective length estimator is not projective invariant.

## 4. A projective curvature estimator

In this section, a projective curvature estimator is defined, assuming that projective lengths between samples are known.

### 4.1. Estimating frames by fitting spirals to data

Given three points-tangents samples, $(\mathbf{x}_{i-1}, \mathbf{x}'_{i-1})$, $(\mathbf{x}_i, \mathbf{x}'_i)$ and $(\mathbf{x}_{i+1}, \mathbf{x}'_{i+1})$, and projective parameters $\sigma_{i-1}$, $\sigma_i$ and $\sigma_{i+1}$, we want to find a logarithmic spiral that passes through the samples with the given projective parameters. We then deduce from the obtained spiral the frame $(\mathbf{Q}_i, \mathbf{Q}'_i, \mathbf{Q}''_i)$ at sample $i$.

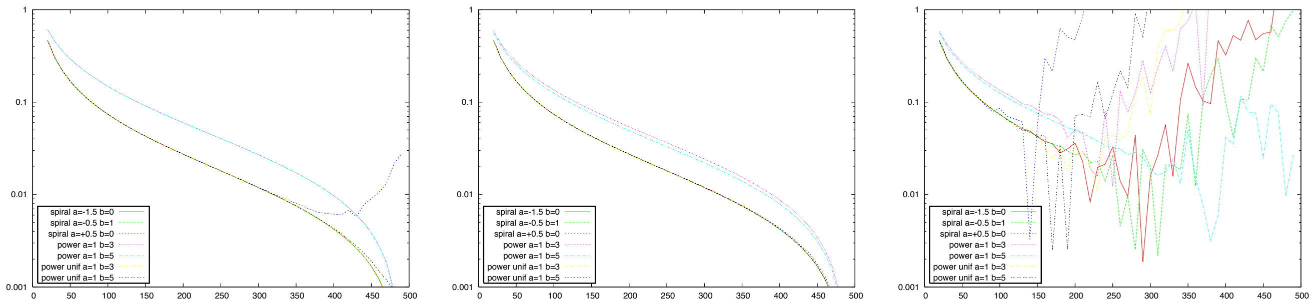


**Figure 4. Same test as in Figure 3, but after applying a projective transformation $T_{test}$: the length estimator (left) and, consequently, the curvature estimator based on it (right) is not projective invariant. Nevertheless, the behavior of the curvature estimator based on exact lengths (middle) is close to that of the original curve, since this estimator is projective invariant.**

**Linear equations of the fitting problem.** Denote by $T_i = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}$ the projective transformation that fits the logarithmic spiral $\mathbf{P}(\sigma)$ to the three consecutive samples. The condition that this spiral passes through the point $\mathbf{x}_j = (x_j, y_j, w_j)$ at $\sigma = \sigma_j$ is given by the equations

$$\begin{cases} AP_x(\sigma_j) + BP_y(\sigma_j) + CP_w(\sigma_j) = \lambda_j \cdot x_j \\ DP_x(\sigma_j) + EP_y(\sigma_j) + FP_w(\sigma_j) = \lambda_j \cdot y_j \\ GP_x(\sigma_j) + HP_y(\sigma_j) + IP_w(\sigma_j) = \lambda_j \cdot w_j \end{cases}$$

where $\lambda_j$ is an unknown parameter. And the condition that the spiral is tangent to $\mathbf{x}'_j = (x'_j, y'_j, w'_j)$ at $\mathbf{x}_j$ is given by

$$\begin{cases} A\, P_x{}'(\sigma_j) + B\, P_y{}'(\sigma_j) + C\, P_w{}'(\sigma_j) = \alpha_j\, x_j + \beta_j\, x'_j \\ D\, P_x{}'(\sigma_j) + E\, P_y{}'(\sigma_j) + F\, P_w{}'(\sigma_j) = \alpha_j\, y_j + \beta_j\, y'_j \\ G\, P_x{}'(\sigma_j) + H\, P_y{}'(\sigma_j) + I\, P_w{}'(\sigma_j) = \alpha_j\, w_j + \beta_j\, w'_j \end{cases}$$

where $\alpha_j$ and $\beta_j$ are unknown parameters (see Figure 1).

**Relaxing one tangency condition.** Assuming that the projective length parameters are known, one has to determine 9 unknown parameters of the projective transformation plus 9 multipliers $\lambda_j, \alpha_j, \beta_j$ for $i = i-1, i, i+1$, from 18 homogeneous equations. Unless one of these equations is redundant, this system has only the trivial solution. However, if a tangency equation is relaxed, the new homogeneous linear system has 15 equations and 16 variables.

We thus drop one tangency condition, which, for the sake of symmetry, is chosen to be at the central point. The corresponding linear system has now rank 15. By fixing the condition $\det(T_i) = 1$, one can find a unique solution $T_i$.

**Projective frame.** The estimated frame at sample $i$ is given by $(\mathbf{Q}_i, \mathbf{Q}'_i, \mathbf{Q}''_i) = T_i \cdot (\mathbf{P}, \mathbf{P}', \mathbf{P}'')$. It is clear that this frame estimator is projective invariant.

## 4.2. Estimating projective curvature

In order to estimate the projective curvature, we need also an estimate of $\mathbf{Q}'''_i$ at sample $i$. Let

$$\mathbf{Q}'''_i = 2\frac{\mathbf{Q}''_{i+1} - \mathbf{Q}''_{i-1}}{\overline{\sigma}_i + \overline{\sigma}_{i-1}}.$$

By decomposing the vector $\mathbf{Q}'''_i$ in the frame, one considers the coefficient $\overline{k}_i$ of $\mathbf{Q}'_i$ as the proposed projective curvature estimator. Thus, we have

$$\overline{k}_i = -\frac{|\mathbf{Q}'''_i, \mathbf{Q}''_i, \mathbf{Q}_i|}{|\mathbf{Q}''_i, \mathbf{Q}'_i, \mathbf{Q}_i|}.$$

This projective curvature estimator is clearly projective invariant. The corresponding estimator for the integral of $k$, $\int k\, d\sigma$, is given by

$$\overline{k} = \sum_{i=3}^{n-2} \frac{|\mathbf{Q}''_{i+1}, \mathbf{Q}''_i, \mathbf{Q}_i| - |\mathbf{Q}''_{i-1}, \mathbf{Q}''_i, \mathbf{Q}_i|}{|\mathbf{Q}''_i, \mathbf{Q}'_i, \mathbf{Q}_i|}.$$

We expect that this estimator is convergent, but until now, we were not able to prove it. Nevertheless, experimental results strongly indicate its convergence (see Figure 3).

**Projective signature.** The above method can also be used to estimate the derivative of the projective curvature. We obtain an estimator $(\overline{k}_i, \overline{k'}_i)$ for the projective signature of the curve writing

$$\overline{k'}_i = \frac{|\mathbf{Q}'''_i, \mathbf{Q}''_i, \mathbf{Q}'_i|}{|\mathbf{Q}''_i, \mathbf{Q}'_i, \mathbf{Q}_i|} - 1.$$

## 5. Implementation and results

**Implementation choices.** For small values of $\sigma$, the samples are almost aligned, and therefore the matrix of the linear system for fitting the spiral to the data has a small determinant. In this context, the method for solving the linear system must be adapted, and we have chosen the LU decomposition. We have also experimented calculating the eigenvector associated with the lowest eigenvalue of the linear system without the tangency relaxation, but the solution was very sensitive to noise.

Another choice that we have done was to use central differences to estimate $Q_i'''$ (see Section 4.2), instead of left or right differences. In fact, in *all* tests that we have performed, the resulting frames with central differences were better than the ones with left or right differences.

**Error measures.** In our model, as in several discrete models [16], curvature is concentrated at vertices. Therefore, the convergence can be better observed on integrals of the curvature rather than on punctual curvature. The same is true for the length. We therefore compared our estimators with analytic ones by computing $\int d\sigma$ and $\int k\, d\sigma$. To test the projective invariance, we consider the same curves before and after a projective transformation $T_{test}$. Here, we have chosen $T_{test} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$.

**Results.** As can be observed in Figures 3 and 4, the length and curvature estimators (left and middle graphs) are convergent, when they are considered independently. However, for small values of $\sigma$, i.e. for a higher number of samples, the numerical instabilities of the length estimator (left graphs) induce a high instability of the combined estimator (right graphs and Figure 2). Figure 5 corroborates the projective invariance of the curvature estimator alone.

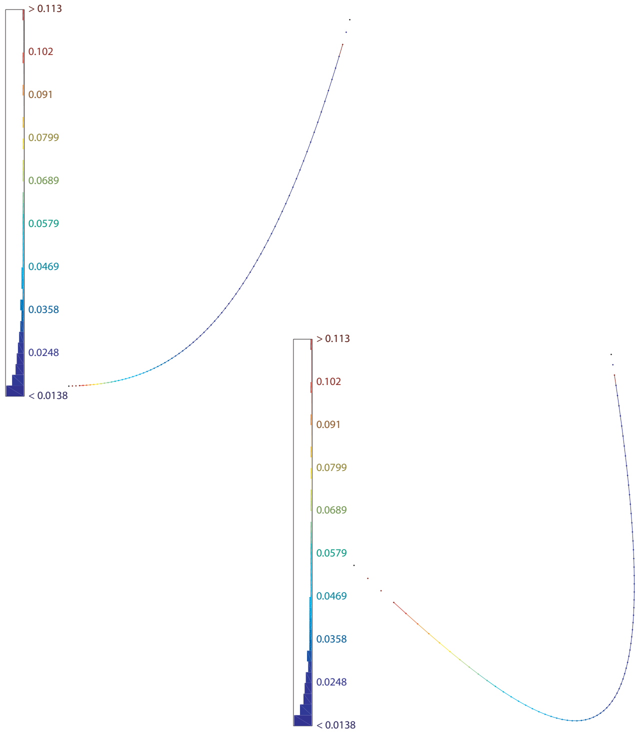


**Figure 5. The estimated projective curvature using the exact analytic lengths on a polynomial curve $(t, t^3, 1)$. Comparing before (left) and after (right) projection $T_{test}$, the error distribution is similar, corroborating the projective invariance of the estimator.**

## 6. Conclusion and future works

In this paper, estimators for the projective length and curvature of a plane curve given by point-tangent samples are proposed. The projective length estimator is based on affine estimators that were proved here to be convergent. The projective length estimator converges in theory and in practice, but it is not projective invariant. We will keep looking for a stable length estimator that is projective invariant.

The projective curvature estimator is based on an estimator of the frame at each sample, which is obtained by fitting logarithmic spirals to the given data. If one assumes that the projective lengths are known in advance, the proposed curvature estimator becomes projective invariant. Its convergence was verified in numerical experiments, but a theoretical proof remains to be done. Taking together both estimators, the experimental results are promising, but also indicate some numerical instability. We plan to continue this work by researching other projective invariant methods for fitting logarithmic spirals to the curve samples. These alternative methods may lead to more robust numerical schemes.

**Acknowledgements.** The authors would like to thank the CNPq for financial support during the preparation of this paper, through projects CNPq 05/2005 and MCT/CNPq 02/2006.

## References


[1] M. Boutin. Numerically invariant signature curves. *International Journal of Computer Vision*, 40(3):235–248, 2000.

[2] M. Boutin. *On Invariants of Lie Groups and their Application to some Equivalence Problems*. PhD thesis, University of Minnesota, 2001.

[3] E. Calabi, P. J. Olver, C. Shakiban, A. Tannenbaum, and S. Hacker. Differential and numerically invariant signature curves applied to object recognition. *International Journal of Computer Vision*, 26(2):107–135, 1998.

[4] E. Calabi, P. J. Olver, and A. Tannenbaum. Affine geometry, curve flows, and invariant numerical approximations. *Advances in Mathematics*, 124:154–196, 1997.

[5] C.A.Rothwell, A.Zisserman, D.A.Forsyth, and J.L.Mundy. Planar object recognition using projective shape representation. *International Journal of Computer Vision*, 16(1):57–99, 1995.

[6] M. Craizer, T. Lewiner, and J.-M. Morvan. Parabolic polygons and discrete affine geometry. In *Sibgrapi*, pages 11–18. IEEE, 2006.

[7] O. Faugeras. Cartan's moving frame method and its application to the geometry and evolution of curves in the Euclidean, affine and projective planes. In *Workshop on Applications of Invariance in Computer Vision*, pages 11–46. Springer, 1994.

[8] O. Faugeras and R. Keriven. Some recent results on the projective evolution of 2D curves. In *International Conference on Image Processing*, volume 3, page 3013. IEEE, 1995.

[9] C. E. Hann. *Recognizing Two planar Objects under a Projective Transformation*. PhD thesis, University of Canterbury, 2001.

[10] R. Fabbri and B. Kimia. High-Order Differential geometry of Curves for Multiview Reconstruction and Matching. In *Energy Minimization Methods in Computer Vision and Pattern Recognition*, pages 645–660. Springer, 2005.

[11] B. Triggs. Camera Pose and Calibration from 4 or 5 known 3D Points. In *International Conference on Computer Vision*, pages 278–284, 1999.

[12] J. Stolfi. *Oriented projective geometry*. Academic Press, Boston, 1991.

[13] P. J.Olver. Joint invariant signatures. *Foundations of Computational Mathematics*, 1(1):3–67, 2001.

[14] S. Lazebnik, Y. Furukawa, and J. Ponce. Projective Visual Hulls. *International Journal of Computer Vision*, 74(2):137–165, 2007.

[15] S. Lazebnik and J. Ponce. The Local Projective Shape of Smooth Surfaces and Their Outlines. *International Journal of Computer Vision*, 63(1):65–83, 2005.

[16] F. Mokhtarian and A. K. Mackworth. A theory for multiscale, curvature based shape representation for planar curves. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 14:789–805, 1992.

## A. Convergence of the affine estimators

Let $\mathbf{x}(s) = (x(s), y(s))$, $-u \le s \le t$, be a convex plane curve parameterized by affine arc-length, with affine curvature $\mu(s)$. Denote $\mu = \mu(0), \nu = \mu'(0)$. Assume that $\mathbf{x}(0) = (0,0)$, $\mathbf{x}'(0) = (1,0)$ and $\mathbf{x}''(0) = (0,1)$. We can write $x(s) = s - \frac{\mu}{6} \cdot s^3 - \frac{\nu}{24} \cdot s^4 + O(s^5)$ and $y(s) = \frac{s^2}{2} + \frac{\mu}{24} \cdot s^4 + -\frac{\nu}{60} \cdot s^5 + O(s^6)$.

Consider that the samples are $\mathbf{x}_i = \mathbf{x}(0)$, $\mathbf{x}_{i-1} = \mathbf{x}(-u)$ and $\mathbf{x}_{i+1} = \mathbf{x}(t)$. Denoting by $\mathbf{z}(t) = (z(t), 0)$ the intersection of the lines defined by $\mathbf{x}'(0)$ and $\mathbf{x}'(t)$, the affine length of the parabola is $L(t) = \sqrt[3]{4z(t)y(t)}$. Let

$$T = \left[ \begin{array}{ll} A(t) & B(t) \\ 0 & D(t) \end{array} \right]$$

be the affine transformation that fixes $(\mathbf{x}(0), \mathbf{x}'(0))$ and takes $(\mathbf{x}_{i+1}, \mathbf{x}'_{i+1})$ to $((t, \frac{t^2}{2}), (1, t))$.

Then direct calculations shows that :

$$\left\{ \begin{array}{lcrrrr} z(t) & = & \frac{1}{2} \cdot t+ & \frac{\mu}{24} \cdot t^3+ & \frac{\nu}{60} \cdot t^4+ & O(t^5), \\ L(t) & = & t+ & & & O(t^5), \\ A(t) & = & 1+ & \frac{\mu}{12} \cdot t^2+ & \frac{\nu}{30} \cdot t^3+ & O(t^4), \\ B(t) & = & -\frac{\mu}{2} \cdot t- & \frac{3\nu}{20} \cdot t^2+ & & O(t^3), \\ D(t) & = & 1- & \frac{\mu}{12} \cdot t^2- & \frac{\nu}{30} \cdot t^3+ & O(t^4). \end{array} \right.$$

The affine curvature estimator is given by

$$\overline{\mu}_i(t, u) = -2 \frac{|\mathbf{q}''(t), \mathbf{q}''(-u)|}{L(t) + L(u)}.$$

where $\mathbf{q}'' = (B, D)$. Direct calculii show that $\overline{\mu}_i(t, u) = \mu_i + \frac{3}{10}(t-u)\nu + O(t^2+u^2)$, which proves the convergence of the affine curvature estimator.

Assuming that the ratios between the affine lengths $t, u, v$ are bounded, we obtain

$$\overline{\mu}_{i+1} - \overline{\mu}_i = \nu_i [u + \frac{3}{10}(t-u) - \frac{3}{10}(u-v)] + O(t^2+u^2+v^2),$$

and so $\overline{\nu}_i = \nu_i + O(t+u+v)$, thus proving the convergence of the derivative of the affine curvature estimator.