Translation and rotation invariant algebraic curve and surface fitting

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Abstract. A method to fit algebraic curves and surfaces using a data independent constraint invariant to rotations and translations is presented. This constraint corresponds to the generalization of the Bockstein constraint to algebraic curves of arbitrary order $p \geq 1$ in $2-D$ space, and algebraic surfaces of arbitrary order $p \geq 1$ in N-dimensional real space, with $N \geq 3$. The fitting is solved using standard eingenvector-eingenvalue results.

1. Introduction

The use of algebraic surfaces has become quite important since it permits modeling complicated forms, as they may arise in problems of medical imaging, robotics, etc., [5]. Algebraic curves and surfaces are an attractive model because they simply use polynomials in the involved variables, and hence they are simple to deal with. An algebraic curve or surface is given implicitly by equating to zero a polynomial on the given variables. The most simple examples are the conic curves: algebraic polynomials of second degree in two variables; and the quadric surfaces: algebraic polynomials of second degree in three variables.

When data is obtained from different locations as in the case of stereo vision, or when the data corresponds to a moving object, one has several image frames corresponding to the same object taken from different view points. In such cases, if we assume that these images correspond to a rigid object, the images correspond to translations and rotations in $3D$ space of the object, [4].

Two models may be assumed for the camera: either the camera performs an orthogonal projection on the image plane, or it performs a perspective projection on the image plane. The first model is adequate if the objects are far away with respect to the focal distance of the lens of the camera. But for both models, the algebraic character of the surface or the curve when projected on the image plane is not altered.

Though there are several different techniques for algebraic curve and surface fitting, one of the most popular is the one that simply minimizes the sum of the squares of the value of the algebraic polynomial for each data point. If the data point lie exactly on the curve, the algebraic polynomial equals zero. Hence the value of the algebraic polynomial measures some sort of "distance" to the curve, when the point does not lies exactly on the curve. Several variants have been proposed so that this "distance" effectively corresponds to the Euclidean distance from the point to the curve. But, the method described previously has the advantage of simplicity, and it is very attractive when thousands of points have to be analyzed, or when the degree of the surface is high. Though the usual problem in computer vision is to deal with planar curves ([3]) and surfaces in 3-D space ([4]), called here algebraic surfaces of dimension two and three respectively, the interest of algebraic surfaces of higher dimension is important in statistical analysis and pattern recognition.

There is a first main reason to introduce a constraint among the coefficients of the algebraic curve. For the error measure introduced above, if all the coefficients are zero, then the error measure is trivially equal to zero. Hence, some form of constraint has to be introduced so that it forces not all of the coefficients to be zero.

Bookstein ([2]) introduced for conic curves, a constraint which guarantees that the obtained coefficients are invariant to rotations and translations of the curves. That is, if the data is fitted using different reference frames, the corresponding curves have compatible coefficients that transforms according to the corresponding translations and rotations of the reference frame. The Bookstein constraint is data independent. Data dependent constraints have the advantage of better fitting algebraic surfaces when the order of the surfaces are not known a priori. The main advantage of the data independent constraint is the simplicity of the algorithm that obtains the surface coefficients.

Such a constraint was later generalized to quadric surfaces independently, by Bookstein and this author, (see references in [5]). In [5], the Bookstein constraint was generalized by this author, to algebraic surfaces of arbitrary number or dimensions through the use of tensor products, in the form of matrix Kronecker products, [1]. In [5] an invariant form of the coefficients of the monomials of degree p (where p is the order of the algebraic surface) was obtained, which generalizes the Bookstein constraint.

The constraint obtained is quadratic in the coefficients of the algebraic surface. Since also the error measure is quadratic in the coefficients of the algebraic surface, a closed form solution in the form of an eigenvalue, eigenvector problem is obtained.

2. Algebraic Surfaces

An algebraic surface in $\mathbf{R}^N$, (the N-dimensional real space), with $N \geq 2$, is given implicitly by a polynomial in the variables $x_1, x_2, \ldots, x_N$ equal to zero. An example of such a surface is:

$$(1) \qquad a x_1^3 x_2^4 x_3^2 + b x_1^2 x_2 x_3 + c x_2 + d = 0$$

The degree of each monomial is defined as the sum of the degrees of each of the variables $x_1, x_2, \ldots, x_N$. In (1) the degree of the first monomial is 9, 4 for the second, 1 for the third, and 0 for the fourth. The degree of the algebraic surface is the highest degree p corresponding to a non-zero coefficient. For the surface defined in (1), the degree of the surface is 9 assuming the coefficient a is non-zero.

The following properties of Kronecker products will be used, [1], where the symbol # denotes the Kronecker product:

$$(2) \qquad \begin{aligned} (A\#B)^T &= B^T\#A^T, \\ (A\#B)(C\#D) &= (AC)\#(BD), \end{aligned}$$

where we assume that the matrices A, B, C and D have matching dimensions, and where the upper-script T denotes matrix transposition.

Hence for a rotation matrix Q (i.e., a matrix such that $Q^TQ = I$), we readily obtain that $Q\#Q\#\cdots\#Q$ is a rotation matrix.

For an arbitrary vector or matrix A, define $A^{\#p}$ as p times the repeated Kronecker product of A,

$$A^{\#p} = \overbrace{A\#A\#\cdots\#A}^{p \text{ times}}.$$

Define $\underline{x}$ as the $N \times 1$ column vector containing the N variables of the curve or surface in $\mathbf{R}^N$, i.e.,

$$\underline{x}^T = (x_1, x_2, \ldots, x_N).$$

Define $\tilde{\underline{x}}$ as the $(N+1) \times 1$ column vector:

$$\tilde{\underline{x}}^T = (\underline{x}^T 1).$$

Call:

$$\text{(3)} \qquad \underline{y} = \underline{x}^{\#p} \qquad \text{and} \qquad \tilde{\underline{y}} = \tilde{\underline{x}}^{\#p}.$$

Hence, $\underline{y}$ is a $N^p \times 1$ column vector, and $\tilde{\underline{y}}$ is a $(N+1)^p \times 1$ column vector.

Then any algebraic surface of degree p in N variables, (or curve of degree p in two variables,) may be written as:

$$\text{(4)} \qquad \underline{a}^T \tilde{\underline{y}} = 0,$$

where $\underline{a}$ is a $(N+1)^p \times 1$ column vector, which contains the coefficients of the monomials in $\tilde{\underline{y}}$.

Note that in the vector $\overline{y}$ many monomials are repeated, and hence, the effective coefficient of a particular monomial is defined as the sum of all the coefficients corresponding to that monomial.

Each monomial of degree $k = j_1 + j_2 + \ldots + j_N$, of the form:

$$x_1^{j_1} x_2^{j_2} \ldots x_N^{j_N},$$

is repeated

$$\text{(5)} \qquad \frac{p!}{j_1! j_2! \ldots j_N! (p-k)!}$$

times in the vector $\tilde{\underline{y}}$, [5].

The total number of *different* monomials of degree k in N variables is, [5]:

$$\text{(6)} \qquad \frac{(k+N-1)!}{k!(N-1)!},$$

and the total number of *different* monomials *up* to degree p in N variables is, [5]:

$$\text{(7)} \qquad \sum_{k=0}^{p} \frac{(k+N-1)!}{k!(N-1)!} = \frac{(p+N)!}{p!N!}.$$

Hence, a conic curve in 2-D space ($p = 2, N = 2$) has 6 different monomials and then 6 effective coefficients, and a quadric surface in 3-D space ($p = 2, N = 3$) has 10 different monomials and then 10 effective coefficients.

Equation (4) may be rewritten as:

$$\text{(8)} \qquad \sum_{j=0}^{p} \underline{a}_j^T \underline{x}^{\#j} = 0,$$

where $\underline{x}^{\#0} = 1$, is a 1×1 scalar.

Under a rotation of coordinates $\underline{x}' = Q\underline{x}$, we obtain:

$$\begin{aligned} \underline{a}_p^T \underline{x}^{\#p} &= \underline{a}_p^T (Q^T \underline{x}')^{\#p} \\ &= \underline{a}_p^T (Q^{\#p})^T \underline{x}'^{\#p} \\ &= a_p'^T \underline{x}'^{\#p}, \end{aligned}$$

where the results of (2) were used, with:

$$\text{(9)} \qquad \underline{a}_p' = Q^{\#p} \underline{a}_p.$$

where $Q^{\#p}$ is an orthogonal matrix, as previously explained. Note that $\underline{a}_p'$ is the vector containing the new coefficients of order p of the surface when referred to the new coordinates $\underline{x}'$.

3. Translation and rotation invariant data independent constraint

The translation and rotation invariant constraint is obtained using the following three observations, ([5]):

1) Under a translation of coordinates, $\underline{x}' = \underline{x} + \underline{t}$, (where $\underline{t}$ is the translation vector), the coefficients of the monomials of order p are not changed. In general, the coefficients of the monomials of order less than p are altered under translations,
2) Under a rotation of coordinates $\underline{x}' = Q\underline{x}$, (where Q is a rotation matrix, i.e., a matrix such that $Q^TQ = I$), the new coefficients of the monomials of order p depend on the rotation matrix Q, and the old coefficients of monomials of order p *only*,
3) If in the vector $\underline{a}$, we take the coefficients corresponding to identical monomials to be equal for each group of monomials, then the vector $\underline{a}'$ obtained after rotation from (9), has the corresponding entries equal, for each of the different groups of monomials.

As an example, consider the case where $p = 3$, and $N = 2$. The monomials of order p in $\underline{y}$ (see (3)) are:

$$x_1^3,\ \ x_1^2x_2,\ \ x_1x_2x_1,\ \ x_1x_2^2,\ \ x_2x_1^2,$$

$$x_2x_1x_2,\ \ x_2^2x_1\ \ \text{and}\ \ x_2^3,$$

together with the corresponding coefficients a_1, a_2, ..., a_8. Hence if we take $a_2 = a_3 = a_5$ and $a_4 = a_6 = a_7$, after rotation it turns that $a_2' = a_3' = a_5'$ and $a_4' = a_6' = a_7'$.

Note that, observation 1) above is readily obtained from (8). Observation 2), is obtained from (9).

As to obtain the constraint, from (9):

$$\underline{a}_p'^T\underline{a}_p' = \underline{a}_p^T(Q^{\#p})^TQ^{\#p}\underline{a}_p = \underline{a}_p^T\underline{a}_p.$$

This result together with observation 1) shows that:

$$\underline{a}_p'^T\underline{a}_p' = \underline{a}_p^T\underline{a}_p \tag{10}$$

is an invariant constraint on the coefficients under rotations and translations of the coordinates.

Equation (10) together with observation 3) above, is the generalization of the Bookstein constraint to curves of degree p in two variables, and surfaces of degree p in three or more variables.

4. Minimization Procedure

The problem to solve is, given N data points in $\mathbf{R}^N$, to try to find the best coefficients of a surface in $\mathbf{R}^N$, of a *given* degree p, using least squares, such that the coefficients found are invariant to translations and rotations of the coordinate system.

Define $\underline{z}$ as the vector obtained from $\underline{\tilde{y}}$ as defined in (3), where all the repeated monomials except one are deleted, and observe that these monomials may be deleted in such a way that the remaining monomials in $\underline{\tilde{y}}$ are ordered in non-increasing degree. The number of entries in $\underline{z}$, is then given by (7). Hence we may split $\underline{z}$ in two vectors $\underline{x}'$ and $\underline{z}''$:

$$\underline{z}^T = (\underline{z}'^T\ \underline{z}''^T), \tag{11}$$

where $\underline{z}'$ contains the monomials of degree p only, and $\underline{z}''$ contains the monomials of degree less than p. The number of entries of $\underline{z}'$ is given by (6) for $k = p$. Define $\underline{b}$ as the vector obtained from $\underline{a}$ where the coefficients corresponding to the deleted monomials of $\tilde{\underline{y}}$ are deleted and the remaining coefficients are multiplied by the factors given in (5). Accordingly $\underline{b}'$ is the vector obtained from $\underline{b}$ corresponding to the scaled coefficients of order p only, and note that none is repeated. Hence we may split the vector $\underline{b}$ in the two vectors $\underline{b}'$ and $\underline{b}''$:

$$\underline{b}^T = (\underline{b}'^T \underline{b}''^T), \tag{12}$$

where $\underline{b}''$ contains scaled coefficients corresponding to monomials of degree less than p. Note that the dimensions of $\underline{z}'$ and $\underline{b}'$ are the same, as well as those of $\underline{z}''$ and $\underline{b}''$.

With these definitions, equation (4) becomes:

$$\underline{b}^T \underline{z} = 0, \tag{13}$$

and the constraint of (10), together with observation 3), becomes:

$$\underline{b}'^T D \underline{b}' = 1. \tag{14}$$

which has been arbitrarily equated to one, and where the matrix D is a diagonal matrix whose coefficients are the *inverses* of the scale factors of the corresponding scaled coefficients.

Given a set of M data points $\underline{x}_i$, for $i = 1, 2, \ldots, M$, first obtain the corresponding vectors z_i as previously explained. In order to find the coefficients that best fit the data points, the least squares problem is, minimize (see (13)):

$$e = \sum_{i=1}^{M} (\underline{b}^T \underline{z}_i)^2, \tag{15}$$

subject to the constraint (14):

$$\underline{b}'^T D \underline{b}' = 1 \tag{16}$$

The minimization of (15) subject to (16) is now solved analogously as in [2], [5], [6], as follows. Using (11) and (12), from (15) one obtains:

$$e = \underline{b}'^T A \underline{b}' + 2\underline{b}'^T B \underline{b}'' + \underline{b}''^T C \underline{b}'', \tag{17}$$

where:

$$A = \sum_{i=1}^{M} \underline{z}'_i \underline{z}'^T_i ,$$

$$B = \sum_{i=1}^{M} \underline{z}'_i \underline{z}''^T_i ,$$

$$C = \sum_{i=1}^{M} \underline{z}''_i \underline{z}''^T_i .$$

Introducing a Lagrange multiplier L in (17) to take in account (16), equivalently minimize the unconstrained problem:

$$E = \underline{b}'^T A \underline{b}' + 2\underline{b}'^T B \underline{b}'' + \\ + \underline{b}''^T C \underline{b}'' - L \underline{b}'^T D \underline{b}'. \tag{18}$$

The solution for (18) is:

$$\hat{\underline{b}}'' = -C^{-1} B^T \hat{\underline{b}}', \tag{19}$$

and

$$(A - BC^{-1}B^T)\hat{\underline{b}}' = LD\hat{\underline{b}}' \tag{20}$$

Matrix C in (19) and (20) has no inverse if the data lies exactly on an algebraic surface of degree less than p, since then, there would exist a vector $\underline{b}''_\varepsilon$ such that:

$$\underline{b}''^T_\varepsilon C \underline{b}''_\varepsilon = 0$$

and then the corresponding vector $\underline{b}'$ could be taken as the null vector, hence satisfying (15) with $e = 0$.

Call p the matrix on the left hand side of (20). Taking an arbitrary vector $\underline{b}_0'$ and correspondingly taking:

$$\underline{b}_0'' = -C^{-1}B^T\underline{b}_0',$$

and substituting in (17), one obtains:

$$e = \underline{b}_0'^T P\underline{b}_0'. \tag{21}$$

Since e is greater or equal than zero for any arbitrary vectors $\underline{b}'$ and $\underline{b}''$, (21) shows that p is a non-negative definite matrix. Note that D is a diagonal positive definite matrix, (see text after (14)).

As to solve (20), let's introduce the following transformation. Eq. (20) may be rewritten as:

$$p\hat{\underline{b}}' = LD\hat{\underline{b}}'$$

Premultiplying by the inverse of the square root of D it is obtained:

$$(D^{-1/2}PD^{-1/2})D^{1/2}\hat{b}' = LD^{1/2}\hat{\underline{b}}'. \tag{22}$$

Call:

$$\underline{q} = D^{1/2}\hat{\underline{b}}', \tag{23}$$

and call:

$$H = D^{-1/2}PD^{-1/2}, \tag{24}$$

so that (22) may be rewritten as:

$$H\underline{q} = L\underline{q}. \tag{25}$$

From (24), note that H is a symmetric and non-negative definite matrix, and then all its eigenvalues are non-negative. Also note from (21) and (15), that P is singular if and only if the data lies exactly on an algebraic surface of degree at most p. Hence if the data does not exactly lie on the curve, the matrices C and P will be non-singular, and then the matrix H will also be non-singular, hence the eigenvalues of H will be strictly positive. Substituting (19) and (20) in (17), and taking in account (16), it is clear that the minimum eigenvalue for H should be chosen.

Hence the procedure is: find the minimum eigenvalue L_m and its corresponding eigenvector $\underline{q}_m$ that solve (25), for which highly efficient algorithms exist, ([6]); normalize $\underline{q}_m$ as to have its Euclidean norm equal to one: that obtain $\hat{\underline{b}}'$ from (23) as:

$$\hat{\underline{b}}' = D^{-1/2}\underline{q}_m,$$

(note that $\hat{\underline{b}}'$ satisfies (16)); and finally obtain $\hat{\underline{b}}''$ from (19).

The vector $\hat{\underline{b}}'$ together with the vector $\hat{\underline{b}}''$, give the coefficients of the algebraic surface that optimize the fit according to the criteria previously explained.

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