	Parameters			
Smoothing	Ramp	0.619		
	Gaussian	3	5	7
	3	0.521	0.695	0.697
	5	0.693	0.694	0.696
	7	0.693	0.707	0.696
	Gamma	3	5	7
	3	0.523	0.694	0.498
	5	0.692	0.693	0.696
	7	0.693	0.707	0.696
	Lognormal	3	5	7
	3	0.519	0.695	0.697
	5	0.692	0.695	0.696
	7	0.693	0.707	0.696
	Beta	3	5	7
	3	0.495	0.699	0.487
	5	0.708	0.710	0.719
	7	0.713	0.727	0.719

Table 3- Mean Square Error of Reconstruction - Real Phantom

4 Conclusions

The experimental results with simulated and real phantoms indicate that it is possible to get improved results with the use of MAP filtering of the projection data in transmission tomography, subject to Poisson noise, as compared to simple ramp filtering of the projections. Furthermore, both the simulated and real tests showed that the use of some "a priori" densities defined over the non-negative real line offered slightly better results on the reconstruction, as compared to the Gaussian distribution. It was also observed that the mean square error with real data is several times greater than with simulated data. This suggests that the real data does not stricly obey the Poisson statistics. At the same time, the good obtained results with the Poisson model demostrates the robustness of the proposed method of reconstruction.

Acknowledgments

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Table 2 displays the results of the mean square error of reconstruction for different filtering procedures, which corroborates the results of the visual analysis.

	Parameters			
	Ramp	0.235		
Smoothing	Gaussian	3	5	7
	3	0.156	0.146	0.144
	5	0.165	0.162	0.160
	7	0.175	0.173	0.173
	Gamma	3	5	7
	3	0.156	0.146	0.146
	5	0.165	0.162	0.161
	7	0.176	0.173	0.174
	Log-normal	3	5	7
	3	0.156	0.146	0.143
	5	0.164	0.161	0.159
	7	0.175	0.171	0.172
	Beta	3	5	7
	3	0.134	0.266	0.263
	5	0.387	0.214	0.157
	7	0.490	0.419	0.387

Table 2 - Mean Square Error of Reconstruction - Simulated Phantoms

3.3 Experimental Results with a Real Phantom

Experiments were also performed with a real phantom of aluminum, under the same conditions as the simulations. The data was collected by a minitomograph scanner for soil science developed at CNPDIA-EMBRAPA [CRUVI90]. Figure 10, 11 and 12 display the results of reconstruction of the real phantom, with the MAP filter (beta and log-normal distributions, 7-point moving average filter and 3-point parameter estimation) and no filtering (using only the ramp filter). Again, superior visual results were obtained with the MAP filters.

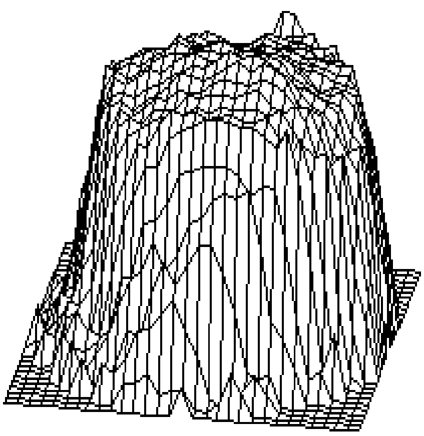

Figure 10- MAP Filtered Reconstruction - Beta distribution - Real Phantom

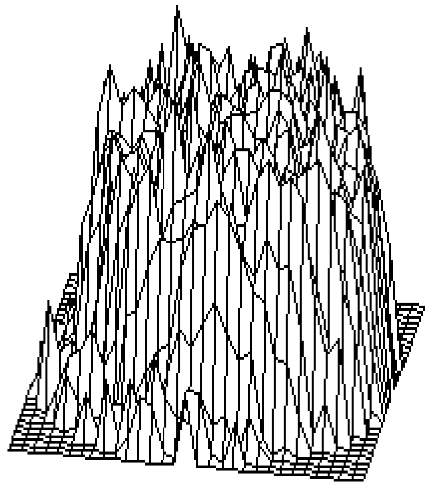

Figure 11- MAP Filtered Reconstruction - Log-normal distribution - Real Phantom

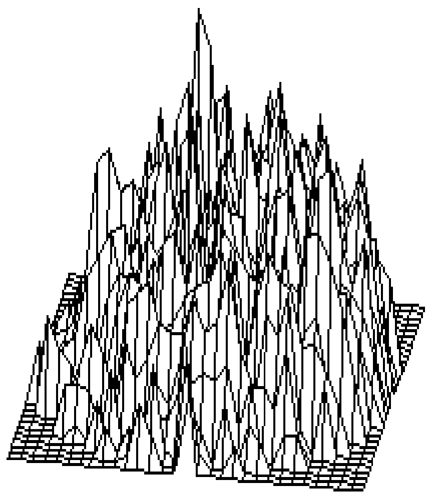

Figure 12- Ramp Filter Reconstruction- Real Phantom

Table 3 displays the results of the mean square error of reconstruction for the real phantom. Again, the error is lower with the MAP filtered projections. although the difference is not as high as in the simulated situation. This is due to the fact that noise in the real case is not strictly Poisson, being greater than what the model predicts.

3.2 Experimental Results with a Simulated Phantom

Simulated projections of 31 points each of a cylindrical aluminum phantom with an ^{241}Am source of gamma rays (1 sec of exposure) were obtained by first determining the rate of counting at each position through the following expression:

$$g_i = g_0 \cdot \exp^{-2\mu\sqrt{R^2 - \left(\frac{iR}{15}\right)^2}} \qquad (-15 \le i \le 15)$$

where g_i is the rate of counting at position i, g_0 is the counting rate at free space (g_0=480), μ is the attenuation coefficient of the aluminum at the ^{241}Am energy (μ=0.747 cm^{-1}) and R is the radius of the cylindrical phantom (R = 3.0 cm). In order to obtain the simulated noisy projections, samples of the Poisson random variables were generated with the rate g_i.

The simulated projections were filtered by the MAP based filtering procedures, using the gaussian, gamma, log-normal and beta distributions, with 3, 5 and 7 points for both the moving average pre-smoothing and the parameter estimation. The reconstruction was performed through a filtering-backprojection method, using the ramp filter. Figures 6, 7, 8 and 9 display the original and reconstructed phantoms with the MAP filtered projections (gamma distribution with 7-point moving average and 3-point parameter estimation and beta distribution with 7-point moving average and 5-point parameter estimation) and no filtering (using only the ramp filter), respectively. One can observe the visually superior results with the MAP filters.

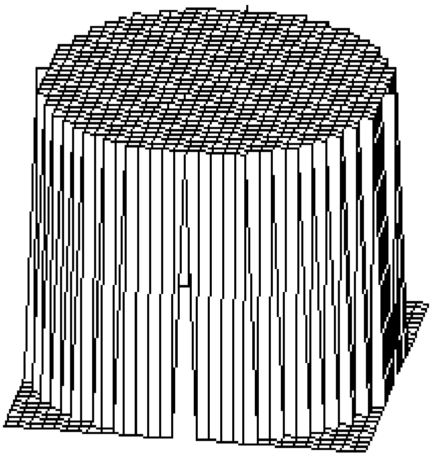

Figure 6- Original simulated phantom

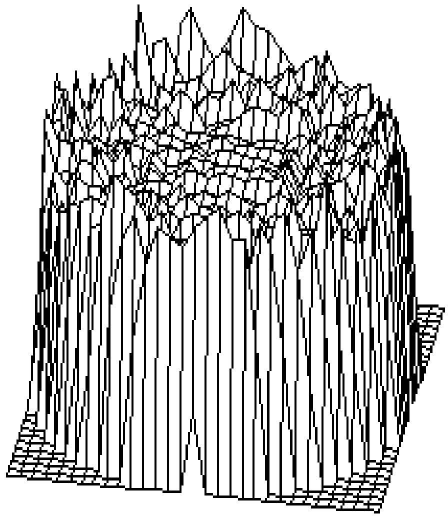

Figure 7- MAP Filtered Reconstruction - Gamma distribution - Simulated Phantom

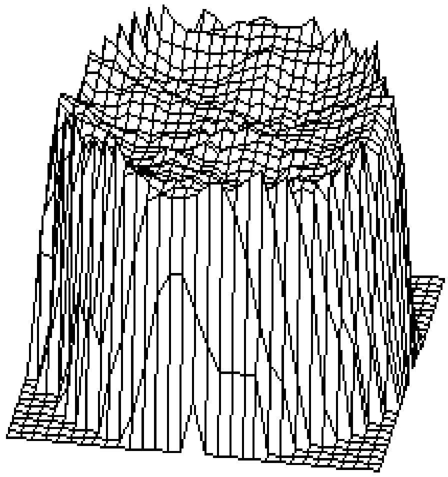

Figure 8- MAP Filtered Reconstruction - Beta distribution - Simulated Phantom

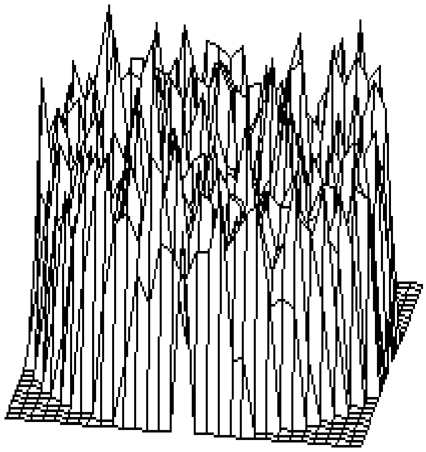

Figure 9- Ramp Filter Reconstruction - Simulated Phantom

In order to apply the MAP based filtering procedures, it is necessary to have access to an approximation to the non-noisy projection. This was done by smoothing the noisy projection by a moving average filter of sizes 3, 5 and 7. The parameter estimation by the method of moments was performed over this smoothed projection by using windows also of size 3, 5 and 7 to collect the sample.

The quantification of the results was done by measuring the root mean square normalized error of estimation (RMSNE), through the expression [FURUI90]:

$$\text{RMSNE} = \sqrt{\sum_{i=1}^{M}(g_i - \hat{g}_i)^2 \Big/ \sum_{i=1}^{M} g_i^2}$$

where $\hat{g}_i$ represents the estimated value.

Figures 3, 4 and 5 below show the filtered projections through the MAP criterion using the exponential, log-normal and gamma densities, respectively. The moving average filter was of size 5 and the parameter estimation was performed over a window of size 3.

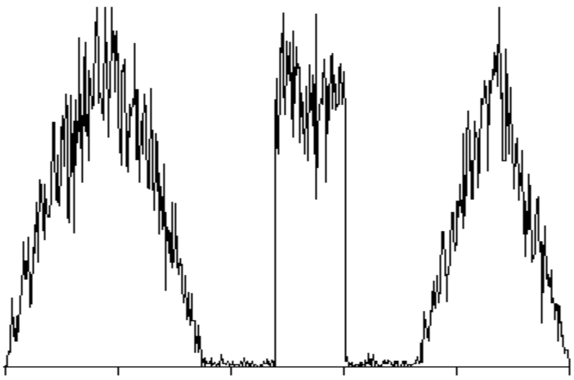

Figure 3 - Filtered simulated projection with exponential prior density

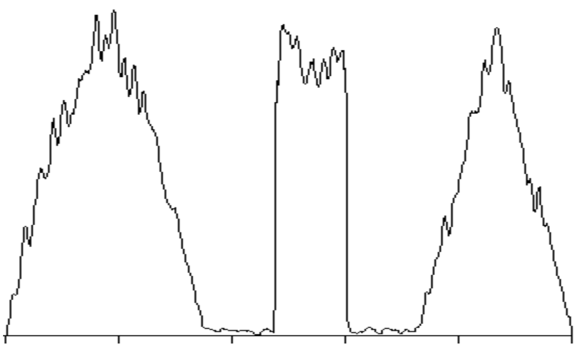

Figure 4 - Filtered simulated projection with log-normal prior density

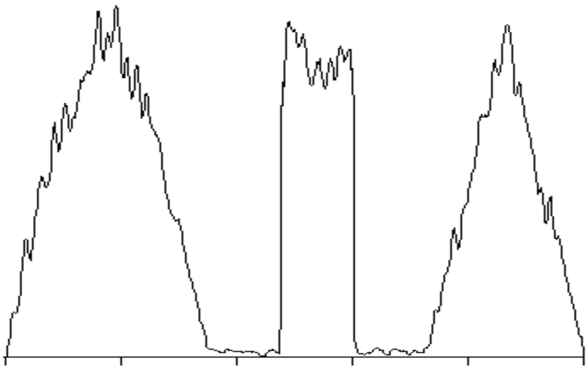

Figure 5 - Filtered simulated projection with gamma prior density

Table 1 gives the results of the RMSNE of estimation for the various "a priori" densities

Densities	**3**	**5**	**7**
Gauss	0.076	0.072	0.069
Gamma	0.076	0.072	0.069
Beta	0.097	0.097	0.093
Lognormal	0.118	0.116	0.116
Chi-square	0.127	0.127	0.126
Rayleigh	0.156	0.156	0.155
Exponential	0.160	0.160	0.159
Noisy projection	0.162		

Table 1 - Comparison of Estimators

One can observe that very little improvement over the noisy projection was achieved with the use of the exponential density. Only the very small values of counting were modified by the filtering process. This is due to the fact that , with the increase of the mean value of the prior distribution, the variance, which is given by the square of the mean value, increases sharply. As a consequence, the prior knowledge in the bayesian estimation is very weak and the MAP estimator is practically given by the maximum likelihood estimator, given by the noisy observation itself. A similar observation is also valid for the Rayleigh and chi-square distributions. A somewhat improved result was obtained with the log-normal and beta distributions, which are parameterized by two independent parameters. The best results were obtained with the gamma and gaussian distributions. The closeness of the results of these two distributions is explained by the fact that the estimated values of the parameter n of the gamma distribution are very high (in the order of hundreds or thousands) and, by the central limit theorem, the gamma density tends to the gaussian density.

2.5 Gamma Density

Given the Gamma "a priori" density

$$f(g) = \frac{\sigma}{\Gamma(\lambda)} (\sigma g)^{\lambda - 1} e^{-\sigma g}, \qquad g > 0$$

the MAP estimator is given by:

$$\hat{g} = \frac{y + (\lambda - 1)}{1 + \sigma}$$

where the parameters λ and σ are estimated by the sample moments m (sample mean), and s^2 (sample variance) :

$$\hat{\lambda} = \frac{m^2}{s^2}$$

$$\hat{\sigma} = \frac{m}{s^2}$$

2.6 Beta Density

Given the Beta "a priori" density

$$f(g) = \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)} g^{\alpha-1}(1-g)^{\beta-1}$$

the MAP estimator is given by:

$$\hat{g} = \frac{1}{2}\Big[(y\Delta + \alpha\Delta + \beta\Delta - \Delta) - \sqrt{(-y\Delta - \alpha\Delta - \beta\Delta + \Delta)^2 - 4(y\Delta^2 + \alpha\Delta^2 - \Delta^2)} \Big]$$

where Δ is the maximum value that is observed in a given projection and the parameters α and β are estimated by the sample moments m (sample mean), and s^2 (sample variance):

$$\hat{\alpha} = \frac{m^2 - m^3}{s^2} - m$$

$$\hat{\beta} = \frac{m^3 - 2m^2 + m + s^2 m - s^2}{s^2}$$

2.7 Log-normal Density

Given the Log-normal "a priori" density

$$f(g) = \frac{1}{\sigma\sqrt{2\pi}} g^{-1} e^{-\frac{1}{2\sigma^2}(\ln g - \mu)^2}$$

the MAP estimator is given by solution of the following transcendental equation:

$$g + \frac{\ln(g)}{\sigma^2} = y + \frac{\mu}{\sigma^2} - 1$$

This equation was solved using the Newton-Raphson method using as initial condition the projection that was smoothed by the moving average filter, where the parameters μ and σ^2 are estimated by the sample moments m (sample mean), and s^2 (sample variance):

$$\hat{\mu} = \log\left(\frac{m^2}{\sqrt{s^2 + m^2}}\right)$$

$$\hat{\sigma}^2 = \log\left(\frac{s^2 + m^2}{m^2}\right)$$

3 Experimental Results

3.1 Filtering of the Simulated Projections

Figure 1 displays the simulated non-noisy projection that was used for testing the proposed algorithms. Each value g_i, i = 1,...,512, represents the average rate of counting in each position for a given angle. The corresponding Poisson noise corrupted projection is represented by y_i, i = 1,...,512 and is displayed in Figure 2.

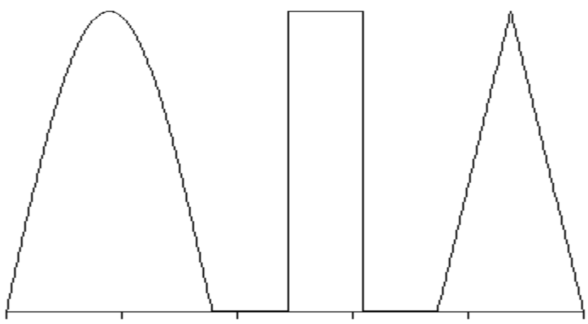

Figure 1 - Original non-noisy simulated projection

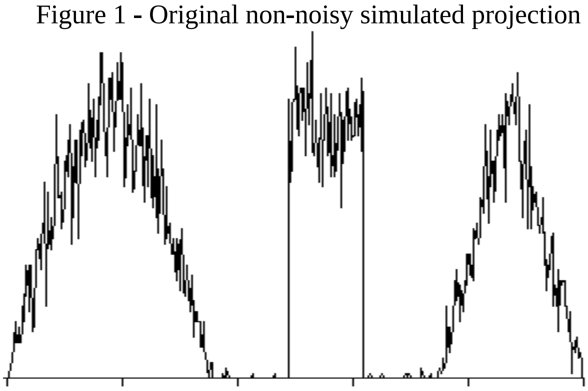

Figure 2 - Noisy simulated projection

2 MAP Estimators for Different "A Priori" Densities

Let us denote by y a given point of the noisy projection and by g the corresponding rate .The adopted model for the Poisson model allow us to express the conditional probability of y, given g, by[GIBR73]:

$$P(y/g) = \frac{e^{-g} g^{y}}{y!}$$

The MAP criterion will maximize the "a posteriori" probability density function f(g/y) and this will depend on the choice of the "a priori" probability density function for g. In the following, we shall express, without proofs, the MAP estimators for different "a priori" density functions. Furthermore, the parameters of these densities will be estimated by the method of moments and the expression of the parameters in terms of the momensts will also be given.

In order to obtain the moments, an initial approximation of the original non-noisy projections has to be obtained. This was done by smoothing the noisy projections by a moving average filter of 3, 5 or 7 points. The sample moments were also obtained by using windows of size 3, 5 or 7.

2.1 Gaussian Density

Given the Gaussian "a priori" density

$$f(g) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{g-\mu}{\sigma}\right)^2}$$

the MAP estimator is given by:

$$\hat{g} = \frac{\mu - \sigma^2 + \sqrt{+\left(\sigma^2 - \mu\right)^2 + 4\sigma^2 y}}{2}$$

where the parameters μ and σ^2 are estimated by the sample moments m (sample mean), and s^2 (sample variance):

$$\hat{\mu} = m = \frac{1}{n}\sum_{i=1}^{n} g_i$$

$$\hat{\sigma}^2 = s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(g_i - \mu_i)^2$$

where n is the size of the window around the estimated point.

2.2 Exponential Density

Given the Exponential "a priori" density

$$f(g) = \sigma\, e^{-\sigma g}, \qquad g>0$$

the MAP estimator is given by:

$$\hat{g} = \frac{y}{(1+\sigma)}$$

where the parameter σ is estimated by the sample moment m (sample mean):

$$\hat{\sigma} = 1/m$$

2.3 Rayleigh Density

Given the Rayleigh "a priori" density

$$f(g) = \frac{g}{\sigma^2} e^{-\frac{g^2}{2\sigma^2}}$$

the MAP estimator is given by:

$$\hat{g} = \frac{-\sigma^2 + \sqrt{\sigma^4 + 4\left(1+\sigma^2 y\right)}}{2}$$

where the parameter σ is estimated by the sample moment m (sample mean):

$$\hat{\sigma} = m\sqrt{\frac{2}{\pi}}$$

2.4 Chi-square Density

Given the Chi-square "a priori" density

$$f(g) = \frac{1}{2^{n/2}\Gamma(n/2)} g^{n/2-1} e^{-g/2}$$

the MAP estimator is given by:

$$\hat{g} = \frac{2y + n - 2}{3}$$

where the parameter n is estimated by the sample moment m (sample mean):

$$\hat{n} = m$$

The Use of MAP Estimation Techniques in the Tomographic Reconstruction of Poisson Noise Corrupted Images

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Abstract The problem of tomographic image reconstruction is important in many areas of applied science and technology. This work presents new methods for the tomographic reconstruction of images with Poisson noise corrupted projections. The Poisson noise comes from the discrete nature of radiation that characterizes the counting process of particles in the projection set. The reconstruction method is performed by first filtering the noisy projections under the Maximum a Posteriori criterion and subsequently reconstructing the images through conventional filtering-backprojection methods using the ramp filter. The "a priori" knowledge is incorporated by using several densities, including the Gaussian and densities defined on the non-negative real line. These densities were used to denote the fact that the rates of counting on the projections are non-negative quantities. Experimental results include the filtering of simulated noisy projections and the reconstruction of simulated and real phantoms scanned with a minitomograph scanner for soil science. The obtained results indicate that, by using the MAP criterion, it is possible to obtain a lower mean square error in reconstruction, as compared to conventional methods of filtering-backrojection, with a very small increase in computational effort.

1 Introduction

The corruption of measurements by Poisson noise occurs very often with nuclear radiation, due to the counting processes that are involved. The degradation of this type of noise is more pronounced under low counts, because we have a low signal-to-noise ratio in this case. In the case of tomography, either by emission or transmission, the projection measurements are affected by the Poisson noise.

Formal statistical methods, like maximum likelihood, implemented by the Expectation-Maximization algorithm [SHEPP82] provide optimum solutions to the problem, but with substantial computational effort.. Traditional methods for tomographic reconstruction, like convolution-backprojection, can cope with the problem, but in an ad-hoc manner. In previous works [FURUI90], [MASC93], a compromise between these two methods was proposed, by first filtering the Poisson noise of the projections, followed by the use of the efficient algorithms associated with convolution-backprojection. In these works, the filtering procedure was either local or global, and the "a priori"knowledge was based on regularization [FURUI92] or the use of the MAP criterion with the Gaussian "a priori"density.

The objective of this work is to explore more realistic models, based on the use of densities that are defined on the non-negative real line. This reflects the fact that counting rates are necessarily non-negative quantities. Therefore,we will present the derivation of the MAP estimation technique for filtering the noisy projections using the following densities: Exponential, Rayleigh, Chi-square, Gamma, Beta and Log-normal., besides the Gaussian. The experimental results include the filtering of simulated projections and the reconstruction of simulated and real phantoms scanned with a minitomograph scanner for soil science.