Fuzzy sets on drawing fair plane curves

MARCELO CORDEIRO BERNARDES*
WU, SHIN-TING
D'OTTAVIANO, ITALA M. L.

Grupo de Computação de Imagens
Departamento de Engenharia de Computação e Automação Industrial
Faculdade de Engenharia Elétrica
CP 6101, Unicamp
13081-970 - Campinas - SP
{mcb,ting}@dca.fee.unicamp.br

Centro de Lógica, Epistemologia e História da Ciencia
CP 6133, Unicamp
13081-970 - Campinas - SP
itala@cle.unicamp.br

Abstract. In this paper an approach for interpolating a given sequence of points by a *fair* plane curve is presented. Since the fairness concept is subjective, a non-classical modeling tool - fuzzy sets - is used. The fuzzyfication and defuzzyfication techniques are outlined. We show that the user can be integrated into the defuzzyfication technique in order to obtain "the most fair" curve. Some results of our implementation are included.

1 Introduction

One of the old geometrical problems that have challenged researchers is to find out a fair curve that interpolates a given sequence of points. There is a variety of interpolation methods, such as Lagrange's, Newton's or spline, but from the designer's point of view the resulting curve may still not be enough fair. Therefore, additional conditions should be established.

Although the definition of fairness is subjective there are some attempts to give quantitative measure of fairness. According to Su and Liu [8] a plane curve is called fair if the following three conditions are satisfied:

- the curve has GC^2-continuity;
- there are no unwanted inflection-points on the curve; and
- its curvature varies in an even manner.

In practice, the proposed solutions for constructing a fair curve from a set of points only minimize the number of curvature extrema. What the classical methods consider as fairness means in fact smoothness. The subjective concept of "unwanted" is understood as no inflection point and "even" as curvature varying almost linearly between two subsequent points. However, we argue whether a fair shape implies necessarily that the resulting curve must not be undulant, since the notion of fairness is extremely subjective and imprecise. Suppose a set of points with a zigzag distribution is given and that the designer desires to have a *fair* curve passing through them. It would be odd to generate a curve without this zigzag shape, which the existing techniques would do, since they do not provide any mean for the designer to adapt the pre-defined objective-function regarding to his requirement. This leads us to look for a new technique that can generate, from the designer's point of view, a fair shape passing through a given set of points, no matter how odd and ambiguous is his concept of fairness.

Formally our interpolation problem can be stated as follows: "Given S=((P_1, *sharper*), (P_2, *smoother*), ..., (P_N, *shaper*)), a sequence of N points $P_i, 1 \leq i \leq N$ and the desired "relative grade of fairness"[1] of an interpolatory plane curve, find this curve."

A proposal to solve this problem is divided in two sub-problems:

- the first estimation of the curve shape as a function of S; and

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[1] explained in Section 4.

- the fine interactive adjustments of the curve shape to conform to the designer's intuitive fairness requirement.

For the first estimation it is interesting to use a curve representation that includes the curvature as its parameter, since curvature reflects directly the "smoothness behavior" of a curve at a point. One way to implement the fine interactive adjustments of the curve shape in an ambiguous fashion seems to be to apply the techniques provided by fuzzy sets. To ensure a perfect matching of these two sub-problems the fair concept must be interpreted by fuzzy sets, which relate the grade of membership of the curve parameter values to this concept (fuzzyfied). From the obtained fuzzy sets the parameter values are computed (defuzzyfied) in order to generate a most likely *fair* curve.

In [1] we proposed an interpolation method for plane open curves. We showed that it generates curve representations with parameters that can be easily associated to the fair concept. In the present paper we show how to fuzzyfy and defuzzyfy such parameters. The paper is organized as follows. Section 2 describes briefly our interpolation method. Section 3 presents some basic concepts about fuzzy sets. Section 4 discusses how we represent the concept of *fairness* as fuzzy sets. Section 5 explains how we can obtain reasonable parameter values from those fuzzy sets. Section 6 shows some obtained results and Section 7 draws some remarks and conclusions.

2 Interpolation method

Let S be the designer given point sequence together with the specification of "how fair" the plane curve that passes through them (grade of "local fairness") should be. The desired (i.e., globally fair) curve C to interpolate the N points in S is looked for.

Our approach is to construct the curve C from an estimated evolute, namely pseudo-evolute (PE), instead of the evolute itself. The estimation of PE is based on the designer given grade of local fairness. From PE the curve C is obtained by determining the so-called radii-vectors. The radii-vectors have the same direction of the PE tangents as the radii of curvature of a curve have the direction of its evolute tangents. So, to obtain C from PE is geometrically analogous to obtain any curve (evolvent) from its evolute [7]. Notice that PE supports the curve global fairness concept, while the radius-vector expresses the curve local fairness concept [1].

To take into account the grade of "local fairness" and to support local manipulations of C, we consider C as a set of pieces C_i and use the evolvent construction to determine each of these pieces. Initially a radius-vector is estimated for each point in S. With these radii-vectors and the points in S, PE is estimated and C is then determined from the interpolated radii-vectors. The directions of the interpolated radii-vectors are given by the PE tangents and their magnitudes are calculated from the linear interpolation of two subsequent estimated radii-vectors.

PE is represented by a set of cubic Bézier curves BP_i as C is represented by a set of C_i. Each Bézier polygon BP_i is associated to a piece C_i. The estimation of BP_i should satisfy the following conditions:

- Its endpoint positions and tangents should agree with the radius-vector magnitudes and directions of the C_i end-points.
- Let P_{i-1}, P_i be two consecutive points in S; C_{i-1} be the piece of C between points P_{i-1},P_i; $\overrightarrow{R_{i-1}}$, $\overrightarrow{R_i}$ be the radii-vectors associated, respectively, to P_{i-1}, P_i; and B_1, B_2, B_3 and B_4 the four control points of the Bézier polygon BP_{i-1}. These control points must satisfy:
  - $B_j = P_i + \alpha_j \overrightarrow{R_i}, \alpha_j \neq 0, j = 1, 2;$ and
  - $B_j = P_{i-1} + \alpha_j \overrightarrow{R_{i-1}}, \alpha_j \neq 0, j = 3, 4.$

  The values of α_j are chosen in such a way that the convexity of the Bézier polygon BP_{i-1} is ensured.

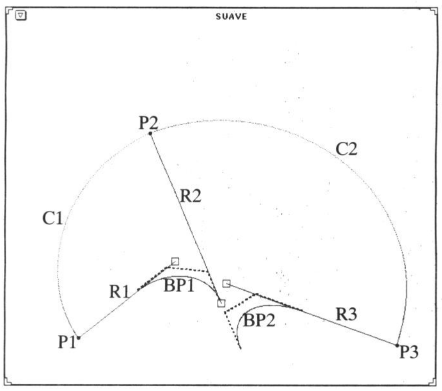

Figure 1: A fair curve passing through three points.

Figure 1 shows a fair curve passing through three given points $\{P_1, P_2, P_3\}$, its corresponding estimated evolute PE (two cubic Bézier curves associated to BP_1 and BP_2) and the estimated radii-vectors $\{R_1, R_2, R_3\}$. The interpolated point coordinates, the radius-vector angles (correspond to radius-vector directions as explained in Section 4) and α_j are given in Table 1.

$P_i(x_i, y_i)$	$\overrightarrow{R_i}$ angle (rad)	α_1	α_2	α_3	α_4
(100,440)	1.30	0.50	0.25	0.50	0.25
(200,150)	1.17	0.50	0.25	0.50	0.25
(550,450)	0.35	0.50	0.25	0.50	0.25

Table 1: Table containing the given points and parameters used in Figure 1

3 Fuzzy sets

Fuzzyness is a type of imprecision inherent to certain classes which do not have defined boundaries. As it is well known in the literature, these classes, the fuzzy sets, that were introduced by Zadeh [9], arise when we look for describing ambiguity, vagueness and ambivalence in mathematical models of empirical phenomena. In particular, the computer simulations of systems of high cardinality, so usual in real world, may require some special non-classical mathematical formulation to deal with the imprecise descriptions. Fuzzy sets, which are classes that admits the possibility of partial membership in them, seem to be an adequate tool for dealing with such kind of problems [5, 6, 9].

Let X denote a space of objects. A fuzzy set A in X is a set of ordered pairs $A = \{(x, \chi_A(x))|x \in X$ and $\chi_A(x) \in [0,1]\}$, with $\chi_A(x)$ being the "grade of membership of x in A" In this work we assume, for simplicity, as in [9], that $\chi_A(x)$ is a number in the interval $[0,1]$, instead of considering its values varying through a more generic algebraic structure [3, 5]. Hence, questions like $x \in X$ may have answers different from yes ($\chi_A(x) = 1$, that is, fullmembership of x) or no ($\chi_A(x) = 0$, that is, nonmembership of x).

The operations OR ($\vee$), AND ($\wedge$) and NOT ($\neg$) between fuzzy subsets A and B on X may be defined in many ways. We adopt the following definitions for these operations [5]:

- OR: $A \cup B = \{(x, Max(\chi_A(x), \chi_B(x)))|x \in X\}$;
- AND: $A \cap B = \{(x, Min(\chi_A(x), \chi_B(x)))|x \in X\}$; and
- NOT: $\neg A = \{(x, 1 - \chi_A(x))|x \in X\}$.

One is fuzzyfying a concept when a fuzzy set is defined to express this concept. Conversely, one defuzzyfies a concept when one representative (*crisp*) value is chosen from the fuzzy set to denote this concept. There are several techniques of fuzzyfication and defuzzyfication. The choice for a special pair (fuzzyfication, defuzzyfication) of techniques depends on the application we are looking for. However, it is important to keep in mind that they are interdependent. For an application, one technique cannot be chosen without considering the strategy of the other.

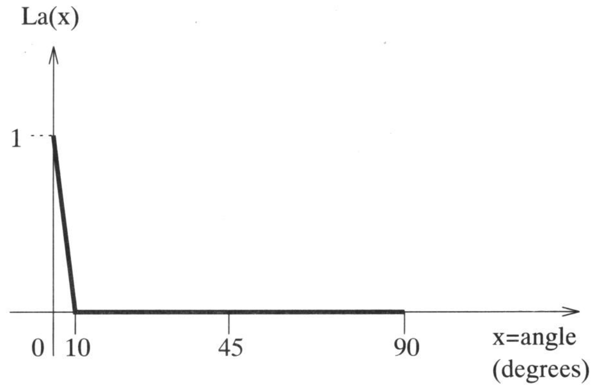

Figure 2: *Almost parallel lines* concept represented as a fuzzy set

As an example, one could define the concept *almost parallel lines* as a fuzzy set A. Given two lines L_1 and L_2, the following classifications of x - the absolute value, in degrees, of the difference between L_1 slope and L_2 slope ($x = |slope(L_1) - slope(L_2)|$) - were assumed:

- x is null. The lines L_1 and L_2 are parallel, so having a unitary membership to *almost parallel lines* concept ($\chi_A(x) = 1, x = 0$);
- x from 0 to 10. The lines L_1 and L_2 are more less parallel, so having a grade of membership to *almost parallel lines* concept that varies linearly from unit to null ($\chi_A(x) = 10 - x, 0 \leq x < 10$); and
- x above 10. The lines L_1 and L_2 are not parallel, so having a nullmembership to *almost parallel lines* concept ($\chi_A(x) = 0, x \geq 10$).

This classification is a fuzzyfication strategy adopted by the designer to express his *almost parallel lines* concept. Figure 2 shows a diagramatic representation of the fuzzy set that describes this concept.

We observe that the grade of membership $\chi_A(x)$ of an object x in A can be interpreted as the degree of compatibility of the predicate associated with A and the object x. It is also possible to interpret $\chi_A(x)$ as the degree of possibility of x being the value of a parameter fuzzyly restricted to A.

Now consider the fuzzy set in Figure 2. Which crisp number could denote actually the fuzzy *almost parallel lines* concept? Suppose that the defuzzyfication technique used by the designer has chosen $x = 3$.

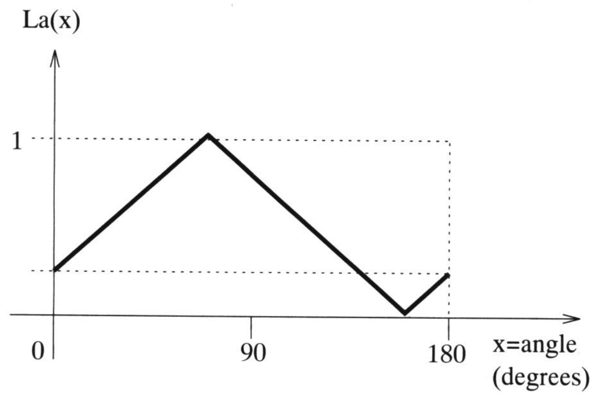

Figure 3: A fuzzy set for the fair concept according to G_1/G_2.

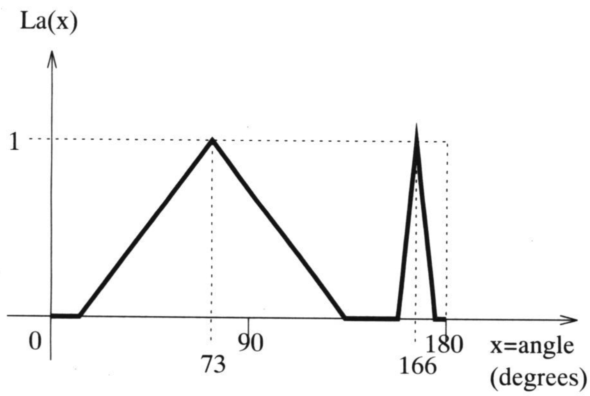

Figure 4: A fuzzy set for the fair concept according to G_3.

So $x = 3$ represents unambiguosly his *almost parallel lines* concept. Other examples can be seen in [5, 6, 9], in special, the example we introduced in [1].

4 Fuzzyfication

Fairness is a very vague concept. Referring to a curve, one may allude to a *fairly* undulant or a *fairly* non-undulant curve or a curve with *fairly* concavity change behavior. In our case, we reduced our problem stated in Section 1 to the construction of a *fair* curve from a set of cubic Bézier curves PE. The control points of these Bézier curves are determined from:

- the points in S;
- radii-vectors associated to these points; and
- α_j.

The points in S are given by the users, so we can only manipulate the radii-vectors and α_j to obtain the desired *fair* curve. Hence, we may fuzzyfy the *fair*

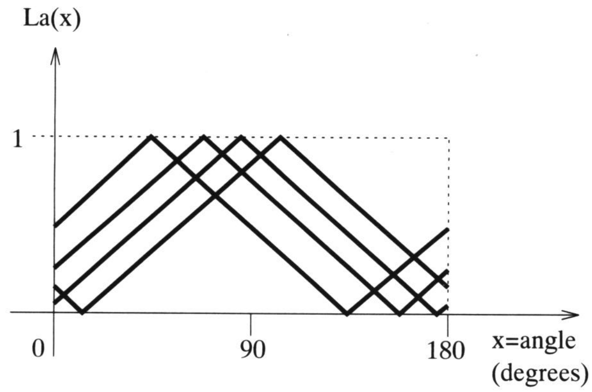

Figure 5: Four fuzzy sets representing the G_1 and G_2 conditions.

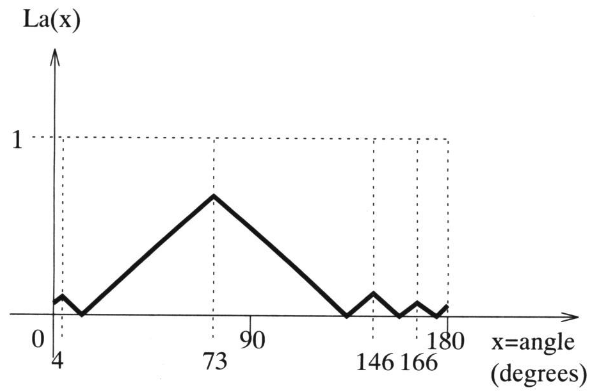

Figure 6: The fuzzy set representing the *fairly undulant* concept for P_i.

curve concept by assigning a grade of membership to the values of these parameters. In this paper we present results of our study on the radii-vectors.

The magnitude of each radius-vector is computed from a designer given SMOOTHER/SHARPER relationship related to the radius-vector of its preceeding point is S. This means that only the relative magnitude of radius-vector matter. Since the relationships between the magnitudes of the radii-vectors are specified by the designer (to convey his expectation of relative fairness behavior) at the points of S, only their directions can vary freely.

In order to reduce the search-space for "good" radius-vector directions, we fuzzyfy each *fair* behavior (and so the *fair* concept) by assigning to these directions a grade of membership. The radius-vector direction is represented by an angle, ranged from 0 to 180 degrees, between the radius-vector and a reference line. Without loss of generality this reference line is the x-axis of the adopted coordinate system. The range [0,180] may be used, instead of [0,360], because only the direction of the radius-vector is necessary in the determination of Bézier polygon, for each C_i. The orientation of the radii-vectors is irrelevant [2].

For assigning a grade of membership to the chosen radius-vector angles we have established the so-called "danger zones" to avoid certain non-fairness shapes. These danger zones allow us to formalize and measure effectively the designer concept of fairness to each *fair* behavior. From them it is possible to devise the range of "values" that the radius-vector angles must not or should assume. Expressing in terms of grade of membership, we say that the values of the radius-vector angles within the danger zones have null or minimal grade of membership to the *fair* concept. The danger zones, within which a radius-vector angle should not be, are determined from some geometrical properties that the cubic Bézier curve should satisfy as [2]:

- G_1 (non-colinear condition): $\overline{R_i} \neq \beta_j(P_i - P_j)$,$\beta_j \neq 0$, $j = i-1, i+1$. If this condition fails, the Bézier polygon (and so C_i) will degenerate to a straight line;
- G_2 (non-parallel condition): $\overline{R_i} \neq \gamma_j(\overline{R_j})$,$\gamma_j \neq 0$, $j = i-1, i+1$. If this condition fails, the Bézier polygon will have $|B_1 - B_2| \ll |B_2 - B_3|$ and/or $|B_3 - B_4| \ll |B_2 - B_3|$, causing oscilations in C_i. If oscilations are desired, this condition can be exploited to get desirable shapes; and
- G_3 (non-intersection condition): $\overline{R_i} \neq \delta_{i-1}(P_i - P_{i-1}) + \epsilon_{i+1}(\overline{R_{i+1}})$,$\overline{R_i} \neq \delta_{i+1}(P_i - P_{i+1}) + \epsilon_{i-1}(\overline{R_{i-1}})$, $\delta_j, \epsilon_j > 0, j = i-1, i+1$. This condition controls concavity changes and oscilations.

Given a relationship GREATER/LOWER among R_{i-1}, R_i and R_{i+1}, which corresponds to the SMOOTHER /SHARPER relationship given by the designer, two situations can occur in G_3:

- G_{31}: the line connecting the intersection point I_{i-1} between radii-vectors R_{i-1} and R_i and the intersection point I_i between radii-vectors R_i and R_{i+1} crosses the convex hull of S an even number of times. It is desired when inflection points and changes in concavity should be avoided; and
- G_{32}: the line connecting the intersection point I_{i-1} between radii-vectors R_{i-1} and R_i and the intersection point I_i between radii-vectors R_i and R_{i+1} crosses the convex hull of S an odd number of times. It is desired when looking for inflection points and changes in concavity.

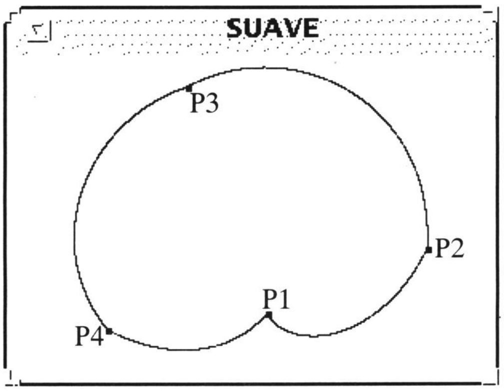

Figure 7: A first estimation of an interpolatory closed curve.

Establishing the conditions of "danger zones" we can now define to each point in S fuzzy sets in the domain of radius-vector angles, [0,180]:

- For G_1: the angle θ_c that validates it has a nullmembership and the angle $\theta_c + \Delta\theta$ a unitary membership to the *fair* concept. A reasonable value for $\Delta\theta$ is 90. The other angles have their grades of membership varying linearly from unit to null as shown in Figure 3.
- For G_2: the classification is similar to the condition G_1 (Figure 3).
- For G_3: the set of angles belonging to the "danger zones" has a nullmembership and the "furthest" angle from the "danger zones" a unitary membership to the *fair* concept. The other angles have their grades of membership varying linearly from unit to null as shown in Figure 4.

From these fuzzy sets we can describe some *fair* behaviors:

- Fairly undulant behavior: for each point P_i in S there are two neighboring points, P_{i-1} and P_{i+1}. The pair (P_{i-1}, P_i) and (P_i, P_{i+1}) must satisfy the non-colinear (G_1) condition and their respective radius-vector pairs (R_{i-1}, R_i) and (R_i, R_{i+1}) the non-parallel (G_2) condition. Therefore, four fuzzy sets are defined (Figure 5). As the four conditions must hold simultaneously, we used an AND operation on these sets to obtain the set F_4 (Figure 6).
- Fairly concavity change behavior: for each connecting point between a pair of consecutive curve pieces (C_{i-1} and C_i) there is one non-intersection condition (G_{31} or G_{32}). The fuzzy set F_2 in Figure 4 represents this behavior.

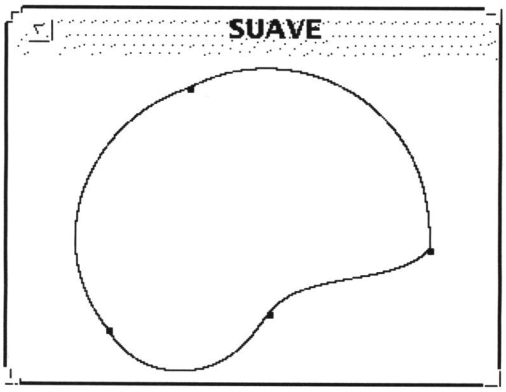

Figure 8: A variant of Figure 7 with $R_1 = 4$.

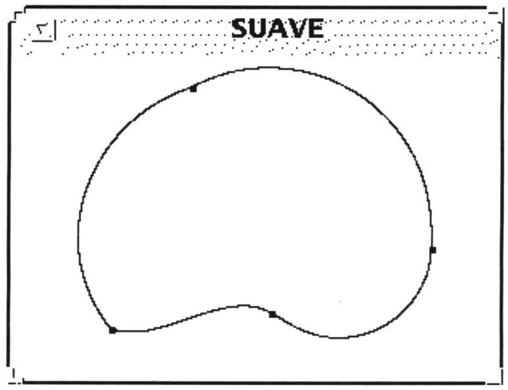

Figure 10: A variant of Figure 7 with $R_1 = 146$.

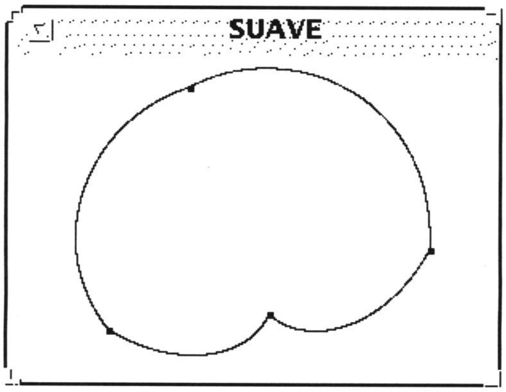

Figure 9: A variant of Figure 7 with $R_1 = 73$.

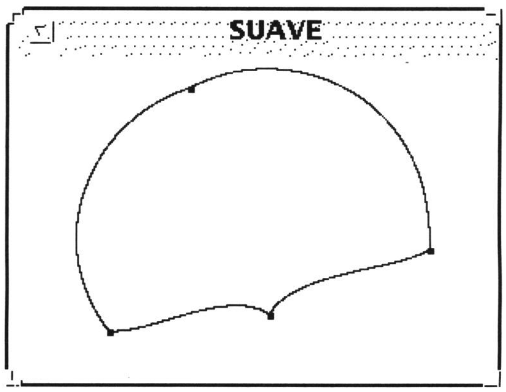

Figure 11: A variant of Figure 7 with $R_1 = 166$.

5 Defuzzyfication

In order to draw a *fair* curve we need a crisp angle value in our interpolation method. So, depending on the kind of *fair* behavior, we should defuzzyfy the corresponding fuzzy set. There is a lot of defuzzyfying techniques [5]. We look for the defuzzyfication technique that support user interactions, so it should satisfy the following conditions:

- to compute more than one crisp value, letting the designer select interactively the most fair curve;
- to compute crisp values not so close from each other, making it easier for the designer to notice the differences between the alternative curves.

In order to reach these conditions, we designed a defuzzyfication technique that chooses the ordered pairs whose grades of membership are local maxima, $M_i, 1 \leq i \leq k$, in the fuzzy set F_k. The designer will then choose one of these pairs based on his subjective *fairness* concept.

As an example let us apply this defuzzyfication technique over the fuzzy set F_4 shown in Figure 6. This fuzzy set expresses the *local fairly undulant behavior* concept to the point P_1 shown in Figure 7. The local maxima, $M_1 = (4; 0.10)$, $M_2 = (73; 0.67)$, $M_3 = (146; 0.13)$ and $M_4 = (166; 0.09)$, were delivered and the corresponding curves are given in Figures 8, 9, 10and 11, respectively.

6 Results

In this section we present some *fairly* undulant interpolatory curves. Figure 7 shows a first estimation for the closed curve obtained from four points. The Figures 12, 13 and 14 are examples of curves resulting from the successive applications of defuzzyfication technique on these points. These applications are controlled by the designer according to his *fair* concept. Observe that different *fair* concepts lead to different "the best" fair curves.

7 Conclusions

The curves generated by the proposed interpolation algorithm were pretty good. We could in the most of the cases adjust easily the shape of the curve until obtaining a *fair* curve. Just in a few situation, we

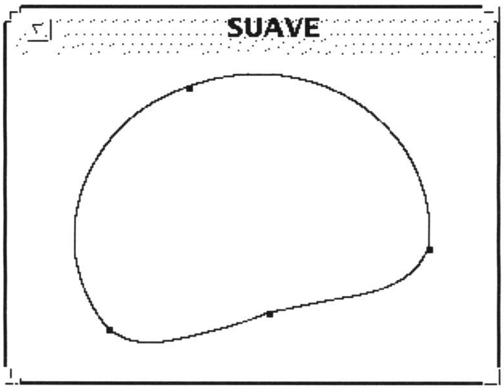

Figure 12: A fair curve with the fair concept that undulations are undesirable.

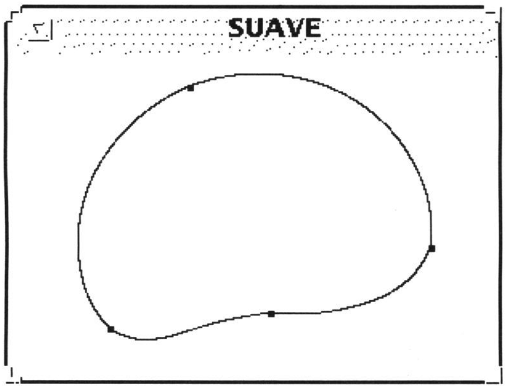

Figure 13: A fair curve with the fair concept that allows undulations.

had difficulties to reach the desired curve. We believe that this difficulty results from the fact that α_j is not adjustable. We expect to overcome this problem by also assigning to the values of this parameter grades of membership to the *fair* concept.

We could improve the proposed defuzzyfication technique by a threshold value, denominated as α-cut R_α, for the local maxima of grades of membership. In this way we can discard angles in the set of local maxima that are improbable to yield *fair* effect. How to obtain a reasonable R_α is still an open problem. These questions are part of our research problem.

8 Aknowledgments

We would like to thank Luiz Henrique Figueiredo and Sueli Costa for their helpful comments and suggestions and Marcelo M. de Gomensaro for providing interaction tools.

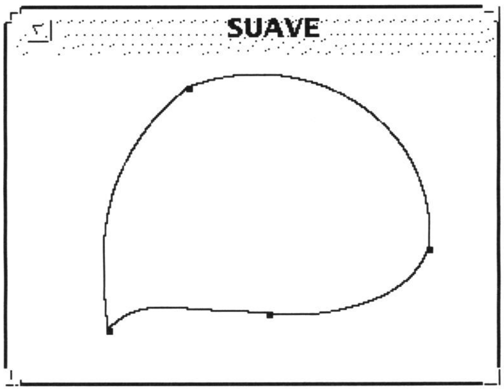

Figure 14: A fair curve with the fair concept that allows cusps.

References

[1] Bernardes, M. C. et al. - An approach to drawing fair plane open curves - Proceedings of the Winter School of Computer Graphics and Visualization'95 - Plzen, Czech Republic (available under: http//:dca.fee.unicamp.br/~mcb/)

[2] Bernardes, M. C. - An approach to drawing fair curves - Master dissertation - DCA-FEE-UNICAMP - in preparation (in Portuguese)

[3] Bolc, L. & Borowik, P. - Many-valued Logics 1: Theoretical Foundations - Springer-Verlag, 1992

[4] do Carmo, M. P. - Differential Geometry of Curves and Surfaces - Prentice Hall, 1976

[5] Kandel, A. - Fuzzy Mathematical Techniques with Applications - Addison-Wesley Publ. Co., 1986

[6] Malinovski, G. - Many-valued Logics - Clarendon Press, 1993

[7] Pogorelov, A. - Geometry - Mir Publishers, 1987

[8] Su Bu-Qing & Liu Ding-Yuan - Computational Geometry - Academic Press, 1990

[9] Zadeh, L. A. - Fuzzy Sets - Information and Control - vol.8 - pp. 338-353, 1965

A Basic geometrical concepts

This appendix briefly presents some fundamentals about intrinsic geometrical properties of a curve [4].

- Curvature: expresses how much the curve "bends". Formally, let $\alpha : I \longrightarrow R^2$ be a curve parametrized by the arc length $s \in I$. The number $\left|\frac{d^2\alpha(s)}{ds^2}\right| = k(s)$ is called the curvature of α at s.

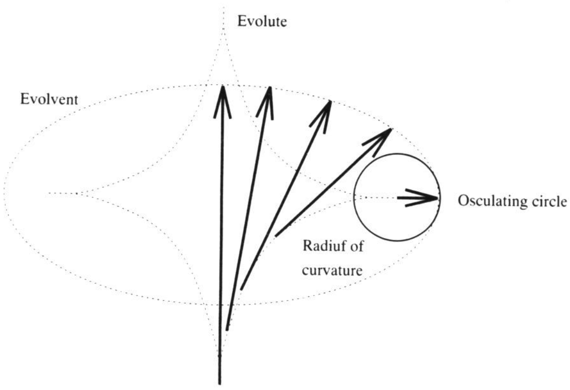

Figure 15: Intrinsic geometrical caracteristics of a curve

- Radius of curvature: is the inverse $R = \frac{1}{k}$ of the curvature (Figure 15).

- Osculating circle: is a second degree approximation of a curve, as the tangent is a first degree approximation. Formally, let $\alpha : I \to R^2$ be a curve parametrized by the arc length s, with curvature $k(s) \neq 0$, $s \in I$. The limit position of the circle passing through $\alpha(s)$, $\alpha(s+h_1)$, $\alpha(s+h_2)$ when $h_1, h_2 \to 0$ is the osculating circle at s, the center of which is on the line that supports the normal vector $n(s)$ and the radius of which is the radius of curvature $\frac{1}{k(s)}$ (Figure 15).

- Evolute: is the geometrical loci of the osculating circles centers. Formally, let $\alpha : I \to R^2$ be a regular parametrized plane curve (arbitrary parameter t), and define normal vector $n = n(t)$ and curvature $k = k(t)$. Assume that $k(t) \neq 0$, $t \in I$. In this situation, the curve $\beta(t) = \alpha(t) + \frac{1}{k(t)}n(t), t \in I$, is called the evolute of α. The curve α is called the evolvent of β (Figure 15). Pogorelov [7] states that it is possible to obtain the evolvent α from its evolute β along with one point of the evolvent (evolvent construction). This may be done as stated (Figure 15):

    1. the evolute tangents are evolvent normals. So, given an evolute, a family of evolvents is determined; and

    2. given a point of the evolvent, we determine within a family which is the desired evolvent.