Invariance for Single Curved Manifold

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Fig. 1. *In this paper, we extend a recent result on invariance under Lambertian model [25]. As a byproduct of our more general result, we may rapidly detect developable regions of a surface: In the (left), there is a 3D mesh; in the (middle) we compute the average discrepancy of the energy level for different "light" vectors; in the (right) we obtain a segmentation, such that dark regions are "close to developable", or developable.*

***Abstract*—Recently, it has been shown that, for Lambert illumination model, solely scenes composed by developable objects with a very particular albedo distribution produce an (2D) image with isolines that are (almost) invariant to light direction change.**

In this work, we provide and investigate a more general framework; and we show that, in general, the requirement for such invariances is quite strong, and is related to the differential geometry of the objects. More precisely, it is proved that single curved manifolds, i.e., manifolds such that at each point there is at most one principal curvature direction, produce invariant isosurfaces for a certain relevant family of energy functions. In the three-dimensional case, the associated energy function corresponds to the classical Lambert illumination model with albedo. This result is also extended for finite-dimensional scenes composed by single curved objects.

***Keywords*-invariance; pattern recognition; developable surface;**

I. Introduction

A broad range of *nuisance factors* makes pattern recognition a hard task to automatize, e.g., noises, measurement failures, to name a few; this is because such factors exhibit *variability*, which leads to an explosion of different patterns to recognize. However, sometimes nuisance factors can be handled by extracting some *sufficient statistic* [17], [23] of the input data, which is invariant to them. This kind of technique is popular in the computer vision community [1], [9], [20], [22], [25], [24]: The measurements are often a two-dimensional image of a three-dimensional scene; and nuisances are, amongst others, illumination conditions, viewpoint changes, occlusions, shadows, quantizations, and noises.

A. Related work

In order to extract sufficient statistics from the input data, naturally, it is necessary to consider some mathematical model of the data source. With this in mind, it is worthwhile mentioning that great models have been produced by the scientific community in order to reproduce the effects of illumination in artificial scenes; see [13] for a comprehensive text. Perhaps the most well-known model is the Lambert illumination model [21]. There are a number of interesting statistics for image data. We can cite the *contours* [18], the *topographic map* [4], [6], or the *attributed Reeb tree* [22] of the image, to name a few. The topographic map, which is the set of all *isolines* of the image, has some advantages: (i) it is contrast-invariant [1]; (ii) it allows for image reconstruction, while not depending on any thresholding parameter; moreover, (iii) several problems have been successfully tackled by using topographic maps [19], [3], [26].

To the best of our knowledge, it has been shown that, for *Lambertian objects*, it is always possible to construct non-trivial viewpoint invariant image statistics [24], whereas general-case (global) illumination invariants essentially do not exist at all [9]. There are a body of literature dealing with more restricted illumination models [1], [19], [2], but only recently, for the Lambertian model, necessary and sufficient conditions on the scene geometry in order to have the topographic map of the image (almost) invariant to the incident light direction have been found [25]. More precisely, it has been proved that solely scenes composed by *developable objects* with a very particular *albedo* distribution produce an image with *isolines* that are (almost) invariant to light direction change.

Contributions: In this work, we theoretically extend Weiss et al.'s work [25] in order to encompass a more general framework: (i) extending the Lambert illumination model in three dimensions by some family of energy functions, (ii) extending three-dimensional objects by finite-dimensional Riemannian manifolds (Section III). We prove that, under certain conditions, if a collection of objects is composed of Riemannian manifolds of Gauss map rank at most 1, then, for almost every parameter, the isosurfaces of the energy function do not vary (Section IV). A direct consequence is, e.g, that the previous change detection algorithm introduced by Weiss et al. [25] also works for different kind of cameras (spherical, cylindrical, etc.), and can be straightforwardly extended and applied to detect wether a manifold is single curved or not, in any finite dimension (Section V).

II. Preliminaries

A. General Notations

Let u be some smooth function, then its gradient is denoted by ∇u. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, then $\langle \mathbf{x}, \mathbf{y} \rangle$ is the classical scalar product. Also $\mathbf{x} \parallel \mathbf{y}$ means that $\mathbf{x} = \lambda \cdot \mathbf{y}$, for some $\lambda \in \mathbb{R} - \{0\}$; in particular, the null vector $\mathbf{0} = (0, \ldots, 0) \parallel \mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^d$. Moreover, $\mathbf{x}^\perp$ denotes the orthogonal complement of the vector space defined by $\mathbf{x}$; i.e., the hyperplane of $\mathbb{R}^d$ passing through the origin, which is orthogonal to $\mathbf{x}$. The normal of some *differentiable manifold* Ω at $\mathbf{x}$ is denoted by $\vec{\mathbf{N}}_\Omega(\mathbf{x})$. Hereafter, we refer to $\mathbb{R}^d - \{\mathbf{0}\}$ as $\mathbb{R}^d_*$ for short, and we assume any manifold to be *pure*, *orientable*, *connected*, *totally bounded*, and *embedded* in $\mathbb{R}^d$, unless mentioned otherwise.

B. Definitions

In this paper, *isosurfaces* are defined as follows [16].

Definition 1 (Isosurfaces)**.** *Let $u : \Omega \to \mathbb{R}$ be some scalar field on a manifold Ω; we suppose that u is defined everywhere. The isosurfaces of u are defined as the boundaries of the connected components of $u^{-1}(\lambda)$, where λ is a constant; see Figure 2.*

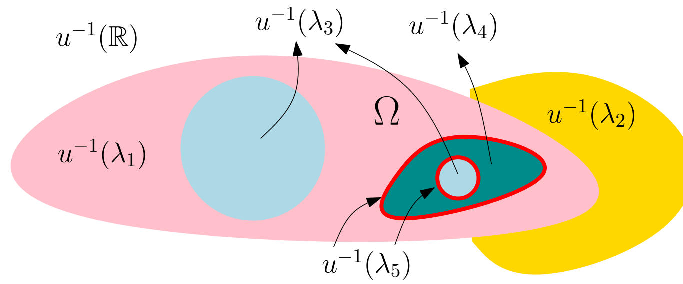


Fig. 2. **Isosurfaces.** *In the figure above, each $u^{-1}(\lambda_i)$, for $\lambda_1, \ldots, \lambda_5 \in \mathbb{R}$, is represented by different colors; and there is a total of seven connected components, each one is an isosurface of u.*

We study in Section III and Section IV, the isosurfaces of a certain Energy Function defined on a manifold Ω_t, which depends on the gradient of a manifold Ω_s (mnemonics $\mathbf{s}$ and $\mathbf{t}$ correspond to the words **s**ource and **t**arget respectively). The following definition clarifies this scenario.

Definition 2 (Energy Function)**.** *Let Ω_s and Ω_t be* Riemannian manifolds *(smooth manifolds equipped with a Riemannian metric) with codimension 1, $\mathbf{v} = (v_1, \ldots, v_d, v_{d+1})$ be a vector in $\mathbb{R}^{d+1}$, $\alpha : \Omega_s \to \mathbb{R}_*$ be a scalar field, and $\psi : \Omega_s \to \Omega_t$ a diffeomorphism, then the Energy Function $E_{\Omega_s,\mathbf{v}} : \Omega_t \to \mathbb{R}$ is defined as:*

$$E_{\Omega_s,\mathbf{v}}(\mathbf{x}) = \Big(\langle \mathbf{h}, \vec{\mathbf{N}}_{\Omega_s}(\psi^{-1}(\mathbf{x})) \rangle + \kappa \Big) \cdot \alpha(\psi^{-1}(\mathbf{x})), \quad (1)$$

where $\mathbf{h} = (v_1, \ldots, v_d)$, and $\kappa = v_{d+1}$, and $d \geq 2$.

In the scenario above, some properties are proved when Ω_s is *single curved* (or even composed of several single curved objects). The following definition makes such a term more precise.

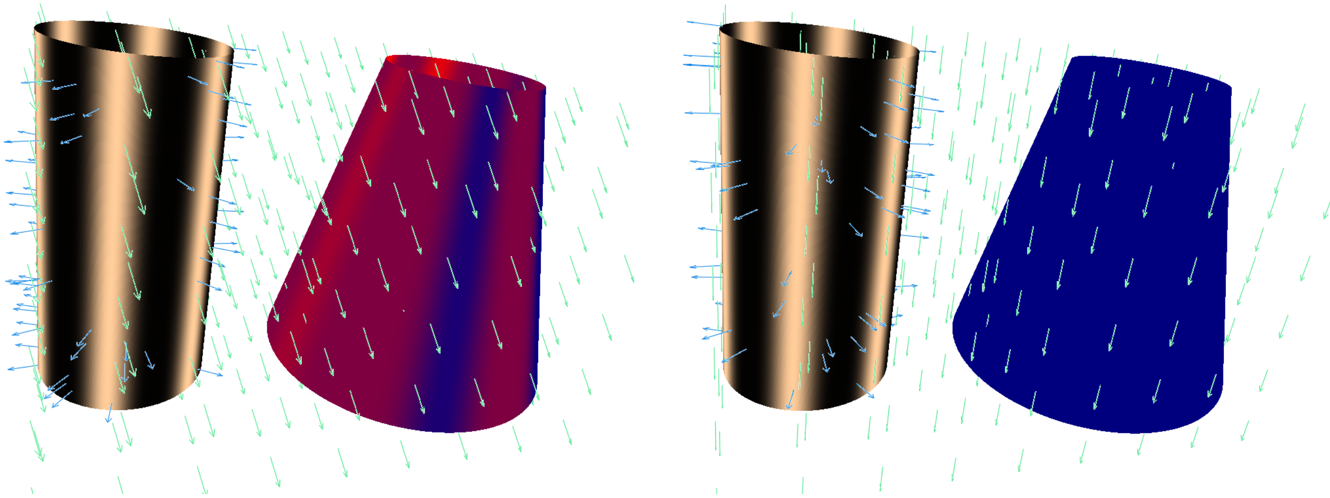
Definition 3 (Single Curved Manifolds)**.** *We call a manifold Ω* single curved *if and only if the rank of its* Gauss map *is everywhere less or equal to* 1.

In other words, Ω is single curved if and only if it has at most one nonzero principal curvature direction at each of its points. Three-dimensional single curved manifolds are simply (piece of) developable surfaces; see Figure 3. In higher dimensions, they are the *osculating scrolls* [12]. (At this point, the reader might want to check [11] for a comprehensive study of continuous differential geometry.)

III. Invariant Isosurfaces: Smooth Case

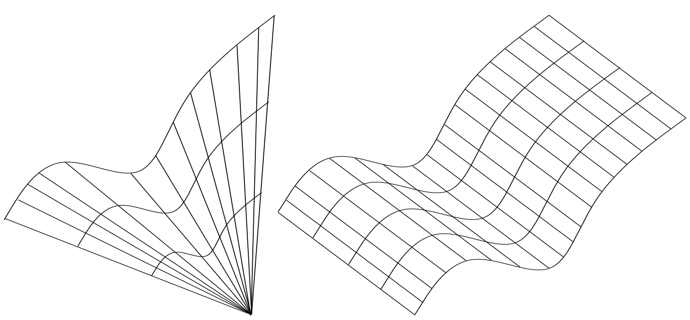
In this section, it is proved that, under some conditions, the isosurfaces of $E_{\Omega_s,\mathbf{v}}$ in Ω_t are the same for almost all $\mathbf{v} \in \mathbb{R}^{d+1}$; the *almost* expression is important (see Figure 4).

The proof is organized as follows: (i) we first prove an equivalence between Ω_s being single curved, and some invariance property on $\nabla E_{\Omega_s,\mathbf{v}}$ (Section III-A); then, (ii) we



(a) separated (b) merged

Fig. 4. **"Almost" is Necessary.** *Eventually, by carefully choosing a vector* $\mathbf{v}$*, isosurfaces can merge; however the vector space, where merging happens, is of Lebesgue measure 0, as we show later in this section.*



(a) cone (b) cylinder

Fig. 3. **Single Curved Surfaces.** *Single curved surfaces in three dimensions are developable surfaces.*

prove that such an invariance property on $\nabla E_{\Omega_s,\mathbf{v}}$ implies that the isosurfaces of $E_{\Omega_s,\mathbf{v}}$ in Ω_t are the same for almost all $\mathbf{v} \in \mathbb{R}^{d+1}$ (Section III-B).

A. Equivalence for Single Curved Manifolds

Theorem 4. *Given the conditions described in Definition 2, the following propositions are equivalent:*

- Proposition 1. *Ω_s is single curved, and α varies only in the (nonzero) principal curvature direction of Ω_s, or freely if Ω_s has no curvature at all.*
- Proposition 2. *$\forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$, $\forall \mathbf{x} \in \Omega_t$, $\nabla E_{\Omega_s,\mathbf{v}_1}(\mathbf{x}) \parallel \nabla E_{\Omega_s,\mathbf{v}_2}(\mathbf{x})$.*

Proof: Since ψ is a diffeomorphism between Ω_s and Ω_t, then Ω_t inherits all the differential properties of Ω_s. Therefore, it is sufficient to show that Theorem 4 holds for when ψ is the identity, and thus, $\Omega_t = \Omega_s$. Also, as κ can be removed by an isometry in $\mathbb{R}^{d+1}$, we drop it without any loss of generality.

Theorem 4 is trivial when Ω_s is a piece of hyperplane, since the normal is constant everywhere. Hereafter, we concentrate our efforts on the case where Ω_s has Gauss map rank equal to 1.

We show now that Proposition 1 and Proposition 2 are equivalent for a fixed point $\mathbf{x} \in \Omega_s$.

Let $T\Omega_s$ denote the tangent space of Ω_s at $\mathbf{x}$, Π the *second fundamental form* of Ω_s at $\mathbf{x}$, and $\vec{\mathbf{N}}$ the normal of Ω_s at $\mathbf{x}$, then, by Eq.(1), we can write $\forall \mathbf{y} \in T\Omega_s$

$$\langle \nabla E_{\mathbf{v}}, \mathbf{y} \rangle = \alpha \cdot \langle \Pi(\mathbf{y}), \mathbf{v} \rangle + \left\langle \vec{\mathbf{N}}, \mathbf{v} \right\rangle \cdot \langle \nabla \alpha, \mathbf{y} \rangle . \tag{2}$$

Consider ***Proposition 1*** $\Rightarrow$ ***Proposition 2*.** By assuming *Proposition 1*, we have that, at $\mathbf{x}$, the rank of Π is 1, and α is constant on the kernel of Π, and then, for some $\lambda \in \mathbb{R}_*$

$$\Pi(\mathbf{y}) = \lambda \cdot \langle \nabla \alpha, \mathbf{y} \rangle \cdot \nabla \alpha. \tag{3}$$

Replacing Eq.(3) in Eq.(2), and solving for $\mathbf{y}$ gives

$$\nabla E_{\mathbf{v}} = \left(\lambda \cdot \alpha \cdot \langle \mathbf{v}, \nabla \alpha \rangle + \left\langle \vec{\mathbf{N}}, \mathbf{v} \right\rangle \right) \cdot \nabla \alpha, \tag{4}$$

which means that $\nabla E_{\mathbf{v}} \parallel \nabla \alpha$ independently of $\mathbf{v}$.

Consider ***Proposition 2*** $\Rightarrow$ ***Proposition 1*.** By assuming *Proposition 2*, we have that there exists $\vec{\mathbf{D}}$, such that, for all $\mathbf{v} \in \mathbb{R}^d$, $y \in T\Omega_s$, and some $\lambda \in \mathbb{R}_*$

$$\begin{aligned} \left\langle \lambda \vec{\mathbf{D}}, \mathbf{y} \right\rangle &= \alpha \cdot \langle \Pi(\mathbf{y}), \mathbf{v} \rangle + \left\langle \vec{\mathbf{N}}, \mathbf{v} \right\rangle \cdot \langle \nabla \alpha, \mathbf{y} \rangle \\ &= \left\langle \Pi(\mathbf{y}) \cdot \alpha + \vec{\mathbf{N}} \cdot \langle \nabla \alpha, \mathbf{y} \rangle , \mathbf{v} \right\rangle . \end{aligned}$$

Now, if y is in the $(d-2)$-dimensional space $T\Omega_s \cap \vec{\mathbf{D}}^{\perp}$, then we have that $\left\langle \lambda \vec{\mathbf{D}}, \mathbf{y} \right\rangle = 0$, and thus

$$\left\langle \underbrace{\Pi(\mathbf{y}) \cdot \alpha}_{\text{tangent to } \Omega_s} + \underbrace{\vec{\mathbf{N}} \cdot \langle \nabla \alpha, \mathbf{y} \rangle}_{\text{normal to } \Omega_s}, \mathbf{y} \right\rangle = 0,$$

which means that $\Pi(\mathbf{y}) \cdot \alpha = 0$ and $\langle \nabla\alpha, \mathbf{y} \rangle = 0$. Because $\alpha \neq 0$, the rank of Π must be 1, furthermore α must be a constant in the kernel of Π.

Finally, varying $\mathbf{x}$ in Ω_s, ends the proof. Note that discontinuities on the rank of the Gauss map are not problematic, since Ω_s is assumed to be an orientable Riemannian manifold, and thus having unique and well-defined normals at any point in Ω_s. ■

Theorem 4 shows an equivalence between a manifold being single curved and the invariance of the Energy function's gradient to direction change, as defined in Definition 2; see Figure 5 for an illustration of the simple geometry of such manifolds.

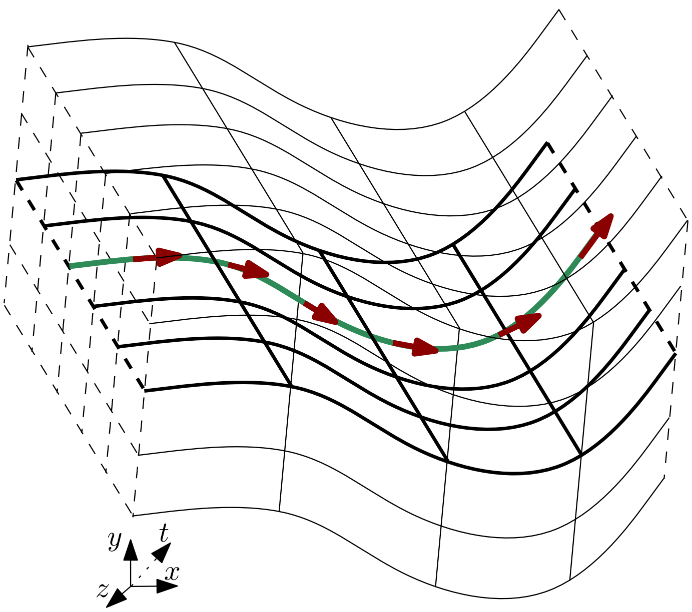


Fig. 5. **Foliation of Ω_s.** *There is a projection on xyz of a 3-dimensional manifold embedded in $\mathbb{R}^4$; it has only one principal curvature direction, and is, thus, a single curved manifold. The generatrices of Ω_s forms a foliation of Ω_s.*

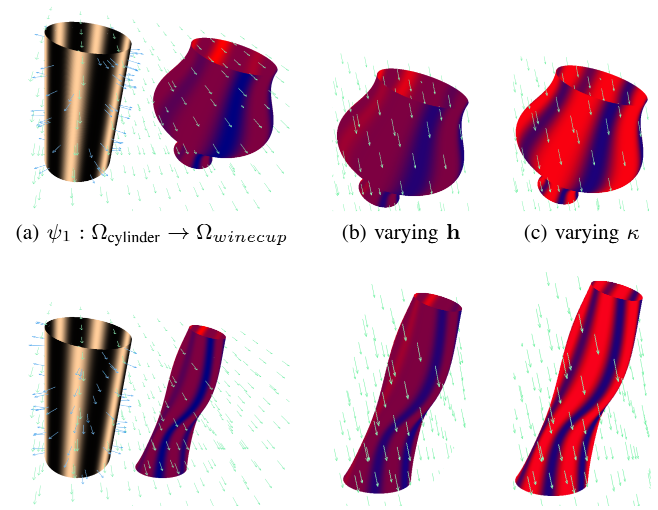
B. Invariance for Single Curved Manifolds

At this point, the reader might want to remind the notations at Section II-A. We additionally introduce the following notations.

Let $\omega \subset \Omega$ be an open set, where Ω is a Riemannian manifold, then $\overline{\omega}$ denotes the *closure* of ω, and $\mathring{\omega}$ or $\mathrm{int}(\omega)$ denote the *interior* of ω defined as the largest open set contained in ω; here, both the notion of closure and open set are related to the *metric space* defined by the Riemannian manifold Ω.

Theorem 5. *Given the conditions described in Definition 2, if Ω_s is single curved, and α varies only in the (nonzero) principal curvature direction of Ω_s (or freely if Ω_s has no curvature at all), then, for almost every $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$, the isosurfaces of $E_{\Omega_s,\mathbf{v}_1}(\mathbf{x})$ are the same as those of $E_{\Omega_s,\mathbf{v}_2}(\mathbf{x})$.*

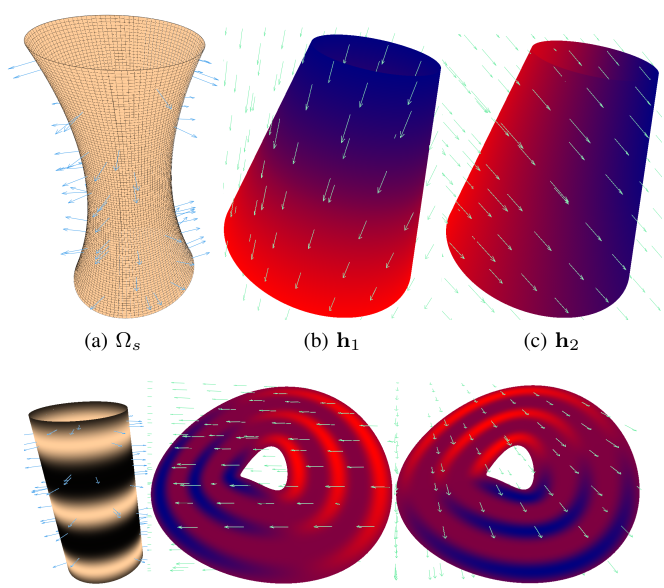
The proof of Theorem 5 is in Appendix-A. Figure 7 and Figure 6 illustrate scenarios where Theorem 5 can be applied, and scenarios where Theorem 5 cannot be applied respectively.



(a) $\psi_1 : \Omega_{\text{cylinder}} \rightarrow \Omega_{winecup}$ (b) varying $\mathbf{h}$ (c) varying κ

(d) $\psi_2 : \Omega_{\text{cylinder}} \rightarrow \Omega_{lady}$ (e) varying $\mathbf{h}$ (f) varying κ

Fig. 6. **Some Valid Scenarios.** *The isosurfaces do not vary for almost every $\mathbf{h}$ and κ.*



(a) Ω_s (b) $\mathbf{h}_1$ (c) $\mathbf{h}_2$

(d) α (e) $\mathbf{h}_1$ (f) $\mathbf{h}_2$

Fig. 7. **Scenarios Failing to Verify the Hypotheses.** *From (a) to (c), the hyperboloid is not developable; and from (d) to (f), the function α is not constant along the generatrices of the cylinder. They do not produce the same isosurfaces.*

Now, we can extend this result with a simple corollary, which is useful in Section IV.

Corollary 6. *Let $\omega \subseteq \Omega_s$, given the conditions described in Definition 2, if Ω_s is single curved and α varies only in the (nonzero) principal curvature direction of Ω_s (or freely if Ω_s has no curvature at all), then, for almost every $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$, the isosurfaces, restricted to $\psi(\omega)$, of $E_{\Omega_s,\mathbf{v}_1}(\mathbf{x})$ and $E_{\Omega_s,\mathbf{v}_2}(\mathbf{x})$ are the same.*

IV. Invariant Isosurfaces: Non-smooth Case

Several objects can be modeled by pieces of single curved (hyper-)surfaces. It turns out, that the same kind of invariance property described in Section III works for those. In this section, we make this point more precise, by defining such piecewise single curved hypersurfaces, and proving their isosurface invariance.

Definition 7 (Piecewise Single Curved Object Space). Ξ *is the space of objects* $\overline{S} = (\overline{\Omega}, \overline{\alpha})$ *such that there exists a finite set* $\{\Theta_i, \alpha_i\}_{i \in I}$ *and* $S = (\Omega, \alpha)$ *which satisfy:*

- $\overline{\Omega}$ *is a compact topological manifold with codimension 1.*
- $\forall (i, j)$, $\Theta_i \cap \Theta_j = \emptyset$.
- $\forall i \in I, \Theta_i$ *is a single curved Riemannian manifold; and* $\cup_{i \in I} \overline{\Theta}_i = \Omega$ *(the closure is with respect to the inherited metric from* Ω*).*
- α_i *varies in the nonzero principal curvature direction of* Θ_i *(or freely if* Θ_i *has no curvature at all).*
- *Finally, we suppose that the normals, and* α *restricted to* Θ_i *admit a limit on the boundary of* Θ_i*. (Of course, the limits do not need to be the same for adjacent pieces.)*

Ξ *is the* piecewise single curved object space *of dimension* d*. An object of* Ξ *is also called a piecewise single curved hypersurface; see Figure 8.*

In what follows, we study the isosurfaces in Ω_t of $E_{\Omega_s, \mathbf{v}}$, when $\Omega_s \in \Xi$; naturally, Ω_s and Ω_t must be isomorphic.

Adopting a diffeomorphism $\psi : \Omega_s \to \Omega_t$ such as the one in Definition 2 becomes more delicate, because of the *sharp features*. We give next, some assumptions on the isomorphism $\phi : \Omega_s \to \Omega_t$, which is called ϕ instead of ψ in order to emphasize differences.

- $\phi : \Omega_s \to \Omega_t$ is an isomorphism;
- ϕ restricted to Θ_i is a diffeomorphism, $\forall i \in I$;

The main result of this section is the following theorem.

Theorem 8. *If* Ω_s *is a finite set of piecewise single curved objects,* Ω_t *is isomorphic to* Ω_s*, and the isomorphism* $\phi : \Omega_s \to \Omega_t$ *satisfies the above-mentioned hypotheses, then, for almost every* $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$*, the isosurfaces of* $E_{\Omega_s, \mathbf{v}_1}(\mathbf{x})$ *and* $E_{\Omega_s, \mathbf{v}_2}(\mathbf{x})$ *are the same.*

The proof of Theorem 5 is in Appendix-B. Figure 8 depicts some $\Omega_s \in \Xi$, some vector $\mathbf{v}$, and the respective isosurfaces on Ω_t.

The next corollary immediately follows from Theorem 8.

Corollary 9. *Let* $\omega \subseteq \Omega_s$*, if* Ω_s*,* Ω_t*, and* $\phi : \Omega_s \to \Omega_t$ *restricted to* ω *satisfy the hypotheses in Theorem 8, then, for almost every* $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$*, the isosurfaces, restricted to* $\phi(\omega)$*, of* $E_{\Omega_s, \mathbf{v}_1}(\mathbf{x})$ *and* $E_{\Omega_s, \mathbf{v}_2}(\mathbf{x})$ *are the same.*

V. Detecting Single Curved Manifold

Some segmentation algorithms in 3D may benefit from good initial guesses [14], [15]. When the segmentation process is related to curvature estimates of the surface, then a reasonable initial guess might be segments of the surface which are "more likely" to be *developable*, i.e. having at most one curvature direction at each point; Figure 3 shows the 3D surface segments of interest for a 3D surface.

One could detect segments of the surface that are likely to be developable by simply estimating discrete curvatures, which can be done fairly well, (e.g., osculating jets [7] or normal cycles [10]) and running some clustering algorithm. However, these estimators are not optimized to detect surface developability, but rather to obtain precise estimates of curvatures in general, which is a harder task. As a consequence, they are quite slow. By using the invariance in Corollary 9, we are able to design a fast algorithm that points out segments of the surface that are likely to be developable without any need for curvature estimation.

The procedure is quite simple, and is based on well-understood contrast equalization algorithms [3], [5]:

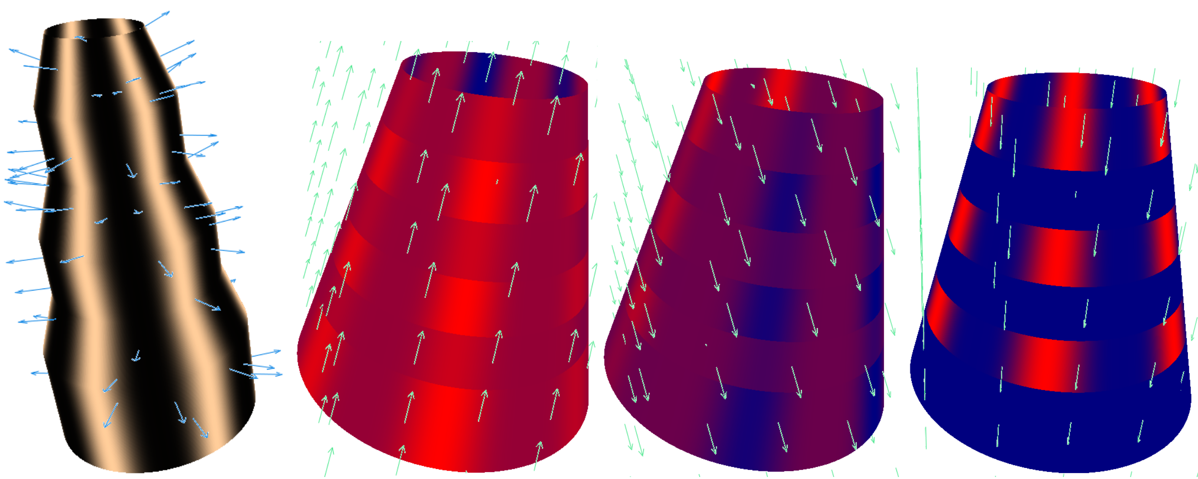
• Let Ω_s be an input 3D surface, represented as a triangulation with M triangles (see Figure 9-left), then we take N unitary vectors $\mathbf{v}_1, \dots \mathbf{v}_N \in \mathbb{R}^4$ at random, and we compute, for each triangle $\mathcal{T} \in \Omega_s$, the energy level $E_{\Omega_s, \mathbf{v}_i}(\mathcal{T})$, $i = 1, \dots, n$; this leaves us with a distribution of energy on the surface of Ω_s. We equalize the distribution, and discretize all values, so the energy value becomes an integer; for short, we denote these discretized energies by $\Delta_i(\mathcal{T})$, corresponding to a discrete $E_{\Omega_s, \mathbf{v}_i}(\mathcal{T})$. (In our implementation, we use integer values ranging from 0 to 7.)

• For each random vector $\mathbf{v}_1, \dots \mathbf{v}_N$, we find $n_1, \dots, n_N$ maximal connected components $\mathcal{C}_{i,k} \subset \Omega_s, k \leq n_i$ respectively, such that $\Delta_i(\mathcal{T})$ is the same for each $\mathcal{T} \in \mathcal{C}_{i,k}$. The connected components above are an approximation of isosurfaces of $E_{\Omega_s, \mathbf{v}_i}$. From Corollary 9, such isosurfaces must be preserved (almost) regardless of light direction change on developable segments of surface. Since the set of vectors such that isosurfaces merge has null measure, random vectors are good choice for invariance. (Of course, perfectly random vectors in an infinite space, such as $\mathbb{S}^3$, are not produced by our algorithm, or even standard computers, however, in practice, the pseudo-random vectors are highly sufficient; indeed, we have got no merging isosurface in any experiment we have done.) With this in mind, we define a discrepancy value $\mathcal{D}_{\mathcal{T}}$ for each triangle $\mathcal{T}$, as follows: Let S be some set of values, then denote the median of S by μ_S, and let $\mathcal{A}_i(S)$ be $\{\Delta_i(\mathcal{T}) \mid \mathcal{T} \in S\}$. The discrepancies $\mathcal{D}_{\mathcal{T}}$ are given by

$$\mathcal{D}_{\mathcal{T}} = \sum_{i=1}^{N} \sum_{j \neq i}^{N} \left| \Delta_j(\mathcal{T}) - \mu_{\mathcal{A}_j(\mathcal{C}_{i,k})} \right|, \mathcal{C}_{i,k} \ni \mathcal{T}. \tag{5}$$

These values represent how the isosurface approximations are similar for different vectors; in an ideal scenario, where the surface is perfectly developable, they would be zero for each pair i, j. $\mathcal{D}_{\mathcal{T}}$ can be computed efficiently in parallel; there is no dependence between triangles and triangles, or triangles and vectors in the computation of the energies.

• After computing $\mathcal{D}_{\mathcal{T}}$ for each triangle, we get a sequence of discrepancies ranging from 0 to 255, forming a distribution of discrepancies on Ω_s; see Figure 9-middle. Let τ be a number ranging from 0 to 1, then finally, we classify as "likely



(a) $\phi : \Omega_{\text{tower}} \to \Omega_{cone}$ (b) $\mathbf{v}_1$ (c) $\mathbf{v}_2$ (d) merging

Fig. 8. **Non-smooth Scenario.** *(a) $\Omega_{tower} \in \Xi$; (b) and (c) show the energy on Ω_{cone} for some parameters; (d) shows isosurfaces that have merged, because of a carefully chosen vector* $\mathbf{v}$.

devlopable" the triangles $\mathcal{T}$, such that $\mathcal{D}_{\mathcal{T}}$ has rank at most τM in the sorted sequence of discrepancies; see Figure 9-right.

Figures 1 and 9 show some results of the procedure above on reasonably sized meshes, ranging from one hundred thousand vertices to around one million and a half vertices. As we can see close to flat, cylindrical, and conical regions tend to have lower discrepancies, while toroidal, spherical, or just doubly curved regions tend to have bigger discrepancies. Each one of these results takes less than half a minute in a MacBook Pro 3,1 equipped with an 2.6 GHz Intel Core 2 processor and 2 GB of RAM, and requires no post-processing at all; see Table I. They can be used as an initial guess for time-consuming procedures that are related to curvature, such as slippage analysis [15]. The complexity of the algorithm is $O(MN \log M + MN^2)$, the lefthand part is due to equalization in the initial step (which is only $O(M)$ after discretization with a simple counting sort).

TABLE I
JET FITTING IS IMPLEMENTED IN CGAL [8], THE PARAMETERS WE USE ARE DEFAULT (-D2 -M2 -A2); TIMINGS ARE EXPRESSED IN SECONDS.

	nb. of facets	N	our work	jet fitting [7]
Pulley (Figure 9)	$\approx$ 500K	30	**17s**	31s
Neptune (Figure 1)	$\approx$ 3M	10	**26s**	196s

The parameters in the above-mentioned procedure are N and τ. Small N might be insufficient to capture the non-developability of some surface segments, and big N might harm the performance of the algorithm. Whereas, small τ might ignore some pretty close-to-developable segments of surface, and big τ might classify wrongly non-developable segments. The values of N and τ in this work are: 10 and 0.7 for Figure 1; 30 and 0.65 for Figure 9. Although, we have fixed the number of possible distinct values for $\Delta_i(\mathcal{T})$ to 7, this is not necessary. However, notice that such a number should not be too large, otherwise too many connected components may have size 1, which is not desirable as $\mathcal{D}_{\mathcal{T}}$ might be artificially reduced.

Finally, Corollary 9 indicates that this same procedure generalizes to any finite dimension, for surfaces of codimension 1. In that case, the patterns are single curved hypersurfaces; i.e., osculating scrolls [12].

VI. Conclusion

In this work, of a more theoretical flavor, we have generalized [25], proving necessary and sufficient conditions for isosurface invariance under a specific family of energy functions in any finite dimension, and we also provided some theoretical and practical applications of our generalizations. Another point is that since the procedure is primordially designed to detect developable parts of an object, its weakness is on false positives; we will tackle this problem in future works. We will also take a look at some other illumination models, such as the ones based on BRDF, in the search for relevant invariances.

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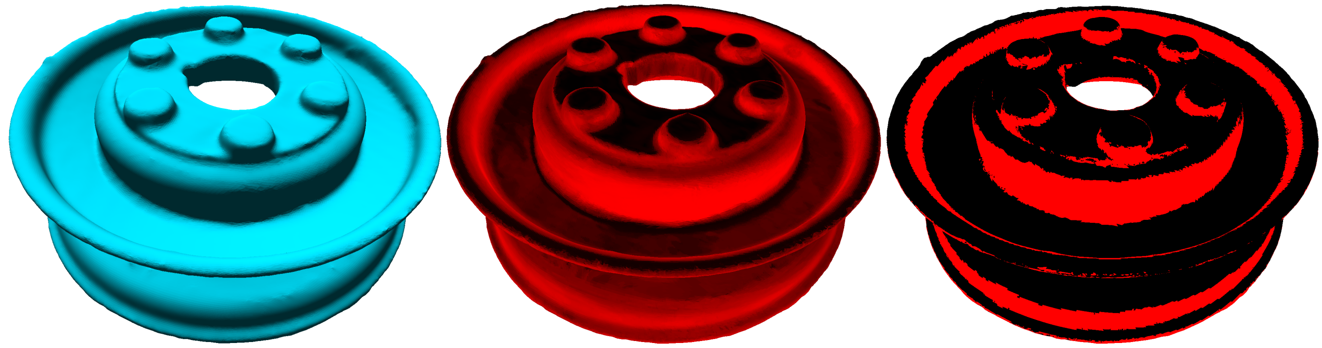

(a) surface (b) distribution (c) segmentation

Fig. 9. **Segmentation.** *In the left, we have the 3D surface; in the middle, we have the distribution of discrepancies ranging from 0 (black) to 255 (red); and in the right, we have the segmentation with $\tau = 0.65$, black pieces are likely to be single curved, whereas red pieces are not; no post processing.*

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Appendix

A. Proof of Theorem 5

In order to prove Theorem 5, we need some kind of *Thom's Tranversality* argument; we follow the same line as that in Weiss et al. [25], and start with the three lemmas below.

Lemma 10. *Let Ω be a Riemannian manifold with codimension 1, $C : \Omega \to \mathbb{R}_*^{d+1}$ be a mapping (no regularity assumption is made), and let $\omega_{\mathbf{v}} = int(\{\mathbf{x} \in \Omega, C(\mathbf{x}) \in \mathbf{v}^{\perp}\})$. For almost every $\mathbf{v} \in \mathbb{R}^{d+1}, \omega_{\mathbf{v}} = \emptyset$; this is with respect to the Lebesgue measure of $\mathbb{R}^{d+1}$, which is denoted by $\mu_{\mathbb{R}^{d+1}}$.*

Proof: Since Ω is a totally bounded metric space, it is *separable*; i.e., there exists a *countable* subset $\Omega_{\mathbb{A}} \subseteq \Omega$ such that every nonempty open subset of Ω contains at least one element of $\Omega_{\mathbb{A}}$. Let $Y = \{\mathbf{v} \in \mathbb{R}^{d+1}, \omega_{\mathbf{v}} \neq \emptyset\}$, and, for each $a \in \Omega_{\mathbb{A}}$, let $Y_a = \{\mathbf{v} \in \mathbb{R}^{d+1}, a \in \omega_{\mathbf{v}}\}$. Y_a is a subset of a hyperplane of $\mathbb{R}^{d+1}$. If it were not the case, there would exist $d+1$ elements of Y_a, $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$, that would form a basis of $\mathbb{R}^{d+1}$. As $C(a) \perp \mathbf{v}_i, \forall i \in \{1, \ldots, d+1\}$, it would mean that $C(a) = \mathbf{0}$, which contradicts $C(a) \in \mathbb{R}_*^{d+1}$. Thus $\mu_{\mathbb{R}^{d+1}}(Y_a) = 0$, and, as $\Omega_{\mathbb{A}}$ is countable, $\mu_{\mathbb{R}^{d+1}}\left(\bigcup_{a \in \Omega_{\mathbb{A}}} Y_a\right) = 0$. Furthermore, $\bigcup_{a \in \Omega_{\mathbb{A}}} Y_a = Y$, since each nonempty open set $\omega_{\mathbf{v}}$ contains an element of $\Omega_{\mathbb{A}}$. Therefore, $\mu_{\mathbb{R}^{d+1}}(Y) = 0$. ∎

Lemma 11. *Let $\omega \in \Omega$ be an open set. Let u_1 and u_2 be two $\mathcal{C}^1$ functions $\Omega \to \mathbb{R}$ such that: $\forall \mathbf{x} \in \Omega,\ \nabla u_1(\mathbf{x}) \parallel \nabla u_2(\mathbf{x})$, $\nabla u_1(\mathbf{x}) \neq \mathbf{0}$ and $\nabla u_2(\mathbf{x}) \neq \mathbf{0}$. Then u_1 and u_2 have the same isosurfaces in ω.*

Lemma 12. *Let u_1 and u_2 be two $\mathcal{C}^1$ functions $\Omega \to \mathbb{R}$ such that $\forall \mathbf{x} \in \Omega, \nabla u_1(\mathbf{x}) \parallel \nabla u_2(\mathbf{x})$. Let $\Omega_1^0 = \{\mathbf{x} \in \Omega, \nabla u_1(\mathbf{x}) = \mathbf{0}\}$, and $\Omega_2^0 = \{\mathbf{x} \in \Omega, \nabla u_2(\mathbf{x}) = \mathbf{0}\}$. The* *following propositions are equivalent:*

- *Proposition 1.* u_1 *and* u_2 *have the same isosurfaces.*
- *Proposition 2.* $\mathring{\Omega}_1^0 = \mathring{\Omega}_2^0$.

The proof of the last two lemmas can be found in Weiss et al. [25] (in their Lemma 2 and Lemma 3), but the notion of closure and open set used in their proofs must be replaced by the closure and open set with respect to the *metric space* defined by Ω.

Now, we have what is needed to prove Theorem 5.

Proving Theorem 5: We assume that Ω_s is single curved, and α varies only in the (nonzero) principal curvature direction of Ω_s (or freely if Ω_s has no curvature at all); from Theorem 4, this implies that $\nabla E_{\Omega_s,\mathbf{v}} = \langle \mathbf{v}, F\rangle \cdot [f_1, \ldots, f_d]$ with $F : \Omega_t \to \mathbb{R}_*^{d+1}$, and $f_1, \ldots, f_d : \Omega_t \to \mathbb{R}$, $\mathcal{C}^0$ mappings, which yields $\forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$, $\forall \mathbf{x} \in \Omega_t$, $\nabla E_{\Omega_s,\mathbf{v}_1}(\mathbf{x}) \parallel \nabla E_{\Omega_s,\mathbf{v}_2}(\mathbf{x})$.

Let $\Omega_t^+ = \{\mathbf{x} \in \Omega_t, [f_1, \ldots, f_d](\mathbf{x}) \neq \mathbf{0}\}$, then this set is open as $f_1, \ldots, f_d$ are $\mathcal{C}^0$. Let

$$\omega_{\mathbf{v}} = \mathrm{int}(\{\mathbf{x} \in \Omega_t^+, \nabla E_{\Omega_s,\mathbf{v}}(\mathbf{x}) = \mathbf{0}\}),$$

then this set is also characterized by

$$\omega_{\mathbf{v}} = \mathrm{int}(\{\mathbf{x} \in \Omega_t^+, F(\mathbf{x}) \in \mathbf{v}^\perp\}).$$

From Lemma 10, we get that for almost every $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$, $\omega_{\mathbf{v}_1} = \omega_{\mathbf{v}_2} = \emptyset$. Now, remark that $\mathrm{int}(\{\mathbf{x} \in \Omega_t, \nabla E_{\Omega_s,\mathbf{v}_1}(\mathbf{x}) = \mathbf{0}\}) = \omega_{\mathbf{v}_1} \cup (\Omega_t - \Omega_t^+)$, and thus, for almost every $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$,

$$\mathrm{int}(\{\mathbf{x} \in \Omega_t, \nabla E_{\Omega_s,\mathbf{v}_1}(\mathbf{x}) = \mathbf{0}\}) = \mathrm{int}(\{\mathbf{x} \in \Omega_t, \nabla E_{\Omega_s,\mathbf{v}_2}(\mathbf{x}) = \mathbf{0}\}).$$

Finally, it suffices to use Lemma 12 to conclude that for almost every $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{d+1}$, $E_{\Omega_s,\mathbf{v}_1}$ and $E_{\Omega_s,\mathbf{v}_2}$ have the same isosurfaces in Ω_t. ■

B. Proof of Theorem 8

Notice that Ω_s and Ω_t are no longer necessarily connected in this scenario. However, this poses no problem at all, as emphasized in the following proposition.

Proposition 13. *In order to prove Theorem 8, we may assume, without any loss of generality, that* Ω_s *is connected (a single piecewise single curved object). This is because* $\phi : \Omega_s \to \Omega_t$ *is assumed to be an isomorphism, and then the image of* ϕ*, for each connected component in* Ω_s*, is non-intersecting.*

Hereafter, we will assume Ω_s to be a single object of Ξ. Naturally, the difficulty involved in proving Theorem 8 now, is handling the boundaries. The next lemma generalizes Weiss et al. [25] (their Proposition 2); it copes with boundaries of finite dimension and with several connected components.

Lemma 14. *Adjacent isosurfaces in* Ω_t *of incident pieces in* Ω_s *merge for almost no* $\mathbf{v} \in \mathbb{R}^{d+1}$.

Proof: Let Θ_s be a piece of Ω_s, $\Theta = \phi(\Theta_s)$, $\mathbf{x}_s \in \Omega_s$, and $\mathbf{x} = \phi(\mathbf{x}_s)$, we call

- $\theta(\mathbf{x}_s, \mathbf{v})$ the isosurface of $E_{\Omega_s,\mathbf{v}}$ containing $\mathbf{x}$;
- $\theta_\Theta(\mathbf{x}_s, \mathbf{v})$, the isosurface $\theta(\mathbf{x}_s, \mathbf{v})$ restricted to Θ.

$(d-k)$-dimensional boundary, $k > 1$. Let, $(\mathbf{x}_{s_{i_1}}, \ldots, \mathbf{x}_{s_{i_k}}) \in \Theta_{s_{i_1}} \times \ldots \times \Theta_{s_{i_k}}$, $(i_1, \ldots, i_k) \in I^k$, and $i_a = i_b \leftrightarrow a = b$, let $\mathbf{x}_{i_j} = \phi(\mathbf{x}_{s_{i_j}})$, $\mathbf{z_c} \in \left[\cap_{j=1}^k \overline{\theta}_{\Theta_{i_j}}(\mathbf{x}_{s_{i_j}}, \mathbf{v})\right]_c$, where $[A]_c$ means some connected component of A, $1 \leq j \leq k$, and $\mathbf{z_c}_s = \phi^{-1}(\mathbf{z_c})$. (Of course, if $\cap_{j=1}^k \overline{\theta}_{\Theta_{i_j}}(\mathbf{x}_{s_{i_j}}, \mathbf{v}) = \emptyset$, there is no boundary incident to these k pieces.) We have, for any $\mathbf{x}_{s_{i_j}}$, $\forall \mathbf{x} \in \theta_{\Theta_{i_j}}(\mathbf{x}_{s_{i_j}}, \mathbf{v})$, $E_{\Omega_s,\mathbf{v}}(\mathbf{x}_{i_j}) = E_{\Omega_s,\mathbf{v}}(\mathbf{x})$, and in particular,

$$\begin{aligned} E_{\Omega_s,\mathbf{v}}(\mathbf{x}_{i_j}) &= \lim_{\substack{\mathbf{x} \to \mathbf{z_c} \\ \mathbf{x} \in \Theta_{i_j}}} E_{\Omega_s,\mathbf{v}}(\mathbf{x}) \\ &= \lim_{\substack{\mathbf{x} \to \mathbf{z_c} \\ \mathbf{x} \in \Theta_{i_j}}} \left(\left(\langle \mathbf{h}, \vec{\mathbf{N}}_s\rangle + \kappa\right) \cdot \alpha\right)(\mathbf{x}). \end{aligned} \tag{6}$$

As $\mathbf{z_c} \in \left[\cap_{j=1}^k \overline{\theta}_{\Theta_{i_j}}(\mathbf{x}_{s_{i_j}}, \mathbf{v})\right]_c$, $\mathbf{z_c} \in [\cap_{j=1}^k \overline{\Theta}_{i_j}]_c$. Since ϕ is an isomorphism, $\mathbf{z_c}_s \in \left[\cap_{j=1}^k \overline{\Theta}_{s_{i_j}}\right]_c$ (the images by ϕ^{-1} of $\Theta_{i_1}, \ldots, \Theta_{i_k}$) and $\lim_{\substack{\mathbf{x} \to \mathbf{z_c} \\ \mathbf{x} \in \Theta_{i_j}}} \mathbf{x} = \mathbf{z_c}$, $j = 1, \ldots, k$. Therefore, we can rewrite Eq.(6) as follows

$$E_{\Omega_s,\mathbf{v}}(\mathbf{x}_{i_j}) = \left\langle \mathbf{h}, \lim_{\substack{\mathbf{x}_s \to \mathbf{z_c}_s \\ \mathbf{x}_s \in \Theta_{s_{i_j}}}} \left(\alpha \vec{\mathbf{N}}_s\right)(\mathbf{x}_s) \right\rangle + \kappa \lim_{\substack{\mathbf{x}_s \to \mathbf{z_c}_s \\ \mathbf{x}_s \in \Theta_{s_{i_j}}}} \alpha(\mathbf{x}_s).$$

Then, for $a, b \in \{1, \ldots, k\}$, $a \neq b$, $E_{\Omega_s,\mathbf{v}}(\mathbf{x}_{i_a}) = E_{\Omega_s,\mathbf{v}}(\mathbf{x}_{i_b})$ if and only if

$$\begin{aligned} &\left\langle \mathbf{h}, \lim_{\substack{\mathbf{x}_s \to \mathbf{z_c}_s \\ \mathbf{x}_s \in \Theta_{s_{i_a}}}} \left(\alpha \vec{\mathbf{N}}_s\right)(\mathbf{x}_s) - \lim_{\substack{\mathbf{x}_s \to \mathbf{z_c}_s \\ \mathbf{x}_s \in \Theta_{s_{i_b}}}} \left(\alpha \vec{\mathbf{N}}_s\right)(\mathbf{x}_s) \right\rangle + \\ &\kappa \cdot \left(\lim_{\substack{\mathbf{x}_s \to \mathbf{z_c}_s \\ \mathbf{x}_s \in \Theta_{s_{i_a}}}} \alpha(\mathbf{x}_s) - \lim_{\substack{\mathbf{x}_s \to \mathbf{z_c}_s \\ \mathbf{x}_s \in \Theta_{s_{i_b}}}} \alpha(\mathbf{x}_s) \right) = 0. \end{aligned} \tag{7}$$

We now consider the following two distinct cases:

- If there is some $a, b \in \{1, \ldots, k\}$, $a \neq b$, such that, $\lim_{\substack{\mathbf{x}_s \to \mathbf{y}_s \\ \mathbf{x}_s \in \Theta_{s_{i_a}}}} (\vec{\mathbf{N}}_s, \alpha)(\mathbf{x}_s) \neq \lim_{\substack{\mathbf{x}_s \to \mathbf{y}_s \\ \mathbf{x}_s \in \Theta_{s_{i_b}}}} (\vec{\mathbf{N}}_s, \alpha)(\mathbf{x}_s)$, then Eq.(7) is verified if and only if $\mathbf{v} = (\mathbf{h}, \kappa)$ lies in a particular hyperplane of $\mathbb{R}^{d+1}$. Such a hyperplane has zero Lebesgue measure.
- Otherwise, $E_{\Omega_s,\mathbf{v}}$ is a constant over $\left[\cap_{j=1}^k \overline{\theta}_{\Theta_{i_j}}(\mathbf{x}_{s_{i_j}}, \mathbf{v})\right]_c$, and then we have a merged single isosurface for this respective boundary. From Corollary 6, we have the invariance for almost every $\mathbf{v} \in \mathbb{R}^{d+1}$ (with respect to the Lebesgue measure).

■

Proving Theorem 8: Since there are countably many boundaries between pieces in Ω_s, from Lemma 14, they account for almost no $\mathbf{v} \in \mathbb{R}^{d+1}$. From Theorem 5, the pieces of Ω_s also account for almost no $\mathbf{v} \in \mathbb{R}^{d+1}$. And hence, Theorem 8 is proved for $\Omega_s \in \Xi$. Finally, Proposition 13 completes the proof for the more general case, when Ω_s is not necessarily connected. ■