# Simulation of Correlated Intensity SAR Images

Oscar H. Bustos[1], Ana G. Flesia[1], Alejandro C. Frery[2]

[1]Facultad de Matemática Astronomía y Física, Ciudad Universitaria, 5000 Córdoba – Argentina
bustos,flesia@mate.uncor.edu
[2]Departamento de Informática, Universidade Federal de Pernambuco, 50732-970 PE – Brazil
frery@di.ufpe.br

**Abstract.** This paper discusses some methods already available for the simulation of correlated heterogeneous targets in SAR images, and extends one of those methods. This new technique is based on the use of a correlation mask and Gaussian random variables, in order to obtain spatially dependent Gamma deviates. Its theoretical properties are presented, along with an algorithm. These Gamma random variables, in turn, allow the obtainment of correlated $\mathcal{K}$ deviates.

## 1 Introduction and definitions

In the synthetic aperture radar (SAR) community the multiplicative model has been widely adopted for the modelling of these images [10]. This model assumes that the value in every pixel, in the intensity format, is the observation of a stochastic process $Z$ defined as the product of two (mutually independent) stochastic processes: $\sigma$ and $Y_I$, where $\sigma$ represents the ground truth and $Y_I$ models the speckle noise

$$Z = \sigma \cdot Y_I.$$

Amplitude format is the square root of the intensity signal. Only intensity data will be treated here.

It is possible to assume that the speckle noise is a white noise process, i.e., formed by independent variables, and that they all obey an exponential distribution with unitary mean. Since this noise is very intense and makes difficult the direct use of the images, it is customary to process the images in order to be able to work with *multilook* data. These data are obtained taking the mean over $n$ (ideally independent) samples of the same image, where from one observation (look) to the next the only possible variation is due to the noise. These samples are obtained in the processing stage and, thus, there is no time elapsed among them.

Calling $Y_{r,I}$ the speckle in each look, and assuming that they all obey a standard exponential distribution, it is well known that the mean $Y_I = n^{-1}\sum_{r=1}^{n} Y_{r,I}$ obeys a Gamma distribution, denoted $Y_I \sim \Gamma(n,n)$ and characterised by the density

$$g_{Y_I}(y) = \frac{n^n}{\Gamma(n)} y^{n-1} \exp(-y/n) \quad y, n > 0.$$

This is a commonly accepted characterisation of the multilook speckle noise in intensity format. In order to derive the law that governs the observed data, it is neccesary to postulate distributions for the ground truth $\sigma$.

A widely used model for the ground truth of heterogeneous and homogeneous targets is the $\Gamma(\alpha,\beta)$ distribution, characterised by the density

$$g_\sigma(\sigma) = \frac{\beta^\alpha}{\Gamma(\alpha)} \sigma^{\alpha-1} \exp(-\sigma/\beta), \quad \sigma, \alpha, \beta > 0,$$

where $\alpha$ is referred to as the shape parameter and $\beta$ as the scale parameter. This distribution, besides showing good fits to a wide range of targets, can be derived from the physical modelling of the way matter and radiation interact in the image formation. This interference phenomenon is present in every image that uses coherent illumination, as is the case of SAR, ultrasound, sonar and laser imaging.

The model for the observed data $Z$, i.e., the product of the mutually independent processes $\sigma$ and $Y_I$, has marginal intensity $\mathcal{K}$ distribution. The correlation introduced in the model of $\sigma$ will induce a certain correlation structure in $Z$. For a detailed discussion of this model and its extension, the reader is referred to [10].

A weakly stationary model will be used for the $\sigma$ field, with a non-trivial correlation structure. This departure from the white noise model requires a precise and unique definition of the family of distributions to be simulated, since the joint density no longer is the product of the marginal densities.

In order to be consistent with the multiplicative model, it is imperative to impose that the marginal distributions obey Gamma laws, but there is no single definition of what a vector of correlated Gamma random variables is. The definition provided in [5, 11] will be adopted here, since it allows the treatment of uncorrelated Gamma random variables as a particular

case of correlated ones.

**Definition 1.1** *The random vector $X'$ obeys a correlated Gamma law if each of its components $X'_i$ marginally obeys a Gamma law.*

**Definition 1.2** *The stochastic process $X$ has a correlated Gamma distribution if each finite subset of $X$ has a correlated Gamma law.*

## 2 Generation of correlated Gamma deviates

Differently from the Gaussian case, where the correlation matrix and the marginal distributions completely specify the joint distribution, these two quantities do not induce an unique joint distribution for correlated Gamma random variables.

The applications that we bear in mind only require the specification of the marginal distributions and the correlation structure. The remaining components required to specify the joint distribution of the process will be induced by the way it is constructed.

Given a set of shape parameters $\alpha_1, \ldots, \alpha_n$, a set of scale parameters $\beta_1, \ldots, \beta_n$, and correlations $\rho_{i,j}$, with $1 \leq i, j \leq n$, it is desired to obtain observations from the random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ such that $X_i \sim \Gamma(\alpha_i, \beta_i)$ and that $Corr(X_i, X_j) = \rho_{i,j}$.

An immediate difficulty that arises with this proposal is that not every set of correlations $\{\rho_{i,j}\}_{1 \leq i,j \leq n}$ is consistent with an arbitrary set of scale parameters $\{\alpha_i\}_{1 \leq i \leq n}$ since the latter set imposes restrictions on the former. Another limitation is that, even with consistent scale parameters and correlations at hand, there might not be a suitable algorithm for the obtainment of the deviates. This is the reason why all the available procedures for the generation of correlated Gamma variables are effective in a restricted domain.

There are simple algorithms that allow the simulation of both positively- and negatively-correlated Gamma random variables for the bivariate case. When more than two random variables are sought, the restrictions are more severe. In the following section the main simulation procedures available for the generation of correlated Gamma random deviates will be presented.

### 2.1 Bivariate Vectors

The methods presented in the sections 2.1.1 and 2.1.2 yield pairs of correlated Gamma random variables, with positive and negative correlation, respectively. They both require a generator of outcomes of independent Gamma random variables.

#### 2.1.1 Trivariate Reduction Method

This method, outlined in [1], allows the generation of a two-dimensional random vector $(X_1, X_2)$ with marginal Gamma distributions with any shape and scale parameters (namely $\alpha_1, \alpha_2$ and $\beta_1, \beta_2$), but imposes the following restriction on the correlation between the components: $0 \leq \rho < \min\{\alpha_1, \alpha_2\}/\sqrt{\alpha_1 \alpha_2}$. Though this is a limitation, this method allows the generation of interesting situations, particularly when the scale parameters are close. In fact, if $\alpha_1 = \alpha_2$ then there is no restriction on the correlation coefficient $\rho$, and if $\alpha_1 = \alpha_2 = 1$ then positively correlated exponential deviates can be obtained. This algorithm is based upon the additive properties of the Gamma distributions. It proceeds as follows:

1. Generate an observation from the random variable $Y_1 \sim \Gamma(\alpha_1 - \rho\sqrt{\alpha_1 \alpha_2}, 1)$;
2. generate a deviate from $Y_2 \sim \Gamma(\alpha_2 - \rho\sqrt{\alpha_1 \alpha_2}, 1)$ which is independent of $Y_1$;
3. generate an outcome of $Y_3 \sim \Gamma(\rho\sqrt{\alpha_1 \alpha_2}, 1)$ which is independent of $Y_1$ and of $Y_2$;
4. return the observation from $X_1 = \frac{1}{\beta_1}(Y_1 + Y_3)$ and $X_2 = \frac{1}{\beta_2}(Y_2 + Y_3)$.

This technique is known as trivariate reduction because the correlated deviates from $X_1$ and $X_2$ are obtained reducing three independent random variables $Y_1$, $Y_2$ and $Y_3$. Note that this algorithm exerts no control over the joint distribution of $(X_1, X_2)$.

Other methods based on the Laplace-Stieltjes transform [7, 8], offer more control over the higher-order moments of the distribution, but they are harder to use in non-trivial situations.

If the simulation of negatively-correlated Gamma variables is needed, the reader can use the following method.

#### 2.1.2 Beta-Gamma Transformation

Some of the authors the have considered the (hard) problem of negatively-correlated Gamma random variables are [6, 12]. Consider the independent identically distributed random variables $\xi_1, \xi_2, \xi_3$ and $\xi_4$ variables obeying a $\Gamma(\alpha, \beta)$ law. Define $X = \xi_1$ and

$$Y = \frac{\xi_2}{X + \xi_2}(\xi_3 + \xi_4).$$

The variable $X$ has $\Gamma(\alpha, \beta)$ distribution while the ratio $\xi_2/(\xi_1 + \xi_2)$ has a Beta distribution with parameters $(\alpha, \alpha)$, and its factor $(\xi_3 + \xi_4)$ is an independent

random variable that obeys a $\Gamma(2\alpha, \beta)$ law. It is possible to see that $Y$ is a $\Gamma(\alpha, \beta)$ distributed random variable, and that there is negative correlation between $X$ and $Y$ because the latter is inversely proportional to the former. It can be proved that this correlation is given by $\rho(X, Y) = -\alpha/(1 + 2\alpha)$ and, therefore, it is bound to the $(-1/2, 0)$ interval.

The advantage of this method is its computational simplicity, though it also requires the use of a generator of Gamma deviates. It can be generalised for higher dimensions, but the distributional properties of the data are hard to obtain unless trivial situations are simulated.

### 2.2 Multidimensional Vectors and Generalised Moving Averages

It is already known that if a moving averages filter of size $L$ is applied to a vector of uncorrelated Gamma random variables, then the result is a vector of correlated Gamma random variables with triangular shaped autocorrelation function, where the shape parameter is multiplied by $L$. More generally, every filter with finite impulse response with binary coefficients will preserve the Gamma marginal distribution and will introduce some correlation structure.

Using this property, Ronning [11] and Blacknell [3] propose methods for the simulation of correlated Gamma deviates. The major problem is the determination of the filters that have to be applied, since they only allow the generation of very simple correlation structures. They are presented in the following sections.

#### 2.2.1 Incidence Matrix Method

This technique was introduced by Ronning [11] as a generalisation of methods for bivariate generation, where only non-negative correlation is obtained (as in section 2.1.1).

Consider $\gamma^{(1)}$ and $\gamma^{(2)}$ vectors of positive constants and dimensions $N^2$ and $M$, respectively, with $M \geq N^2$. Assume that $\xi^{(1)} = (\xi_1^{(1)} \dots \xi_{N^2}^{(1)})$ and $\xi^{(2)} = (\xi_1^{(2)} \dots \xi_M^{(2)})$ are independent random vectors such that $\xi_i^{(1)} \sim \Gamma(\gamma_i^{(1)}, 1)$ and $\xi_j^{(2)} \sim \Gamma(\gamma_j^{(2)}, 1)$ for every $1 \leq j \leq M$, and $1 \leq i \leq N^2$. Then the covariances matrices of $\xi^{(1)}$ and $\xi^{(2)}$ are, respectively,

$$\begin{aligned} \Gamma_1 &= \text{Diagonal}(\gamma_1^{(1)}, \dots, \gamma_{N^2}^{(1)}) \\ \Gamma_2 &= \text{Diagonal}(\gamma_1^{(2)}, \dots, \gamma_M^{(2)}). \end{aligned}$$

Consider $\mathbf{T}$ an *incidence matrix*, i.e., $\mathbf{T}$ is a $N^2 \times M$ matrix such that $\mathbf{T}_{i,j} \in \{0, 1\}$. Defining the vector $\eta = \xi^{(1)} + \mathbf{T}\xi^{(2)}$ it is possible to prove that

1. the covariance matrix of $\eta$ is $\Sigma = \Gamma_1 + \mathbf{T}\Gamma_2\mathbf{T}^t$;
2. if $\alpha = (\alpha_1, \dots, \alpha_{N^2})$ is the diagonal of the covariance matrix of $\eta$, then $\alpha = \gamma^{(1)} + \mathbf{T}\gamma^{(2)}$;
3. denoting every element of $\Sigma$ by $\sigma_{i,j}$, then
$$\begin{aligned} \sigma_{i,i} &= \gamma_i^{(1)} + \sum_{k=1}^{M} \mathbf{T}_{i,k}\gamma_k^{(2)}\mathbf{T}_{i,k}, \\ \sigma_{i,j} &= \sum_{k=1}^{M} \mathbf{T}_{i,k}\gamma_k^{(2)}\mathbf{T}_{j,k} \end{aligned}$$
4. every component $\eta_i$ has $\Gamma(\alpha_i, 1)$ distribution for every $1 \leq i \leq N^2$.

In this manner, the vector $\eta = (\eta_i)_{1 \leq i \leq N^2}$ has a correlated Gamma distribution with means given by $\alpha = (\alpha_1, \dots, \alpha_{N^2})$ and covariance matrix $\Sigma$.

In order to introduce different scale parameters, consider the positive numbers $\beta_1, \dots, \beta_{N^2}$ and $\mathbf{B} = \text{Diagonal}\,(1/\beta_1, \dots, 1/\beta_{N^2})$. If $\Psi = \mathbf{B}\eta$ it is immediate that the marginal distributions are $\psi_k \sim \Gamma(\alpha_k, \beta_k)$. It is also possible to see that the correlation between $\psi_k$ and $\psi_j$ is the same as the correlation between $\eta_k$ and $\eta_j$.

With these results, in order to generate Gamma correlated deviates with a certain correlation structure it is neccesary to derive the incidence matrix $\mathbf{T}$ as well as $\Gamma^{(1)}$ and $\Gamma^{(2)}$ in order to have $\eta$ with the desired correlation matrix $\Sigma$. An algorithm for this is as follows:

1. Define $M = N^2(N^2 - 1)/2$, $\Sigma$ the correlation matrix and $B$ the diagonal matrix with the desired scale parameters.
2. Choose $\gamma^{(2)}$ a vector of constants and $\mathbf{T}$ an incidence matrix such that
$$\begin{aligned} \sigma_{i,i} &= \gamma_i^{(1)} + \sum_{k=1}^{M} \mathbf{T}_{i,k}\gamma_k^{(2)}\mathbf{T}_{i,k}, \\ \sigma_{i,j} &= \sum_{k=1}^{M} \mathbf{T}_{i,k}\gamma_k^{(2)}\mathbf{T}_{j,k}. \end{aligned}$$
3. Generate random deviates from $\xi_1^{(2)}, \dots, \xi_M^{(2)}$, i.e., independent samples from $\Gamma(\gamma_i^{(2)}, 1)$ distributions.
4. Define $\gamma_i^{(1)} = \sigma_{i,i} - \sum_{k=1}^{M} \mathbf{T}_{i,k}\gamma_k^{(2)}\mathbf{T}_{i,k}$, for every $1 \leq i \leq N^2$.
5. Obtain samples of $\xi_1^{(1)}, \dots, \xi_{N^2}^{(1)}$, i.e., independent deviates from $\Gamma(\gamma_i^{(1)}, 1)$ random variables.

6. Return $\Psi = \mathbf{B}(\xi^{(1)} + \mathbf{T}\xi^{(2)})$.

This method cannot yield Gamma random variables with negative correlation, and the shape parameters are imposed by the desired correlations. Moreover, it is often numerically unstable.

### 2.2.2 Moving Average Filter Method

This technique, due to Blacknell [3], is based on the use of moving average filters over independent Gamma random variables. The analysis of the filter is performed using the moment generating function of the result.

If $X \sim \Gamma(\alpha, \beta)$ then its moment generating function is $M_X(s) = E(\exp(Xs)) = (1 - \frac{1}{\beta}s)^{\alpha}$.

Consider $\mathbf{X} = (X_1, \ldots, X_N)^t$ part of a weakly stationary process; if for every $\mathbf{s} = (s_1, \ldots, s_N)$, with $|\mathbf{s}| < \delta$, holds that $M_{\mathbf{X}}(\mathbf{s}) = E\prod_{j=1}^{N} \exp(X_j s_j) < \infty$, then $M_{\mathbf{X}}$ is called moment generating function of $\mathbf{X}$. Note that $M_{X_i}(s_i) = M_{\mathbf{X}}((0, \ldots, s_i, \ldots, 0)^t)$ for every $s_i$, therefore if marginal Gamma distributions are sought for each $X_i$ with shape and scale parameters $\alpha$ and $\beta$ the following conditions must be verified:

$$\begin{aligned} M_{\mathbf{X}}((s, 0, \ldots, 0)^t) &= M_{\mathbf{X}}((0, s, \ldots, 0)^t) \\ &= M_{\mathbf{X}}((0, \ldots, s)^t) \\ &= (1 - \frac{1}{\beta}s)^{\alpha}. \end{aligned}$$

We also have that $E(X_i X_j) = \frac{\partial}{\partial j}\left[\frac{\partial}{\partial i} M_{\mathbf{X}}(\mathbf{0})\right]$ and, therefore, if the correlation $\rho_j$ is desired at lag $j$ then it must be imposed that

$$\rho_j(\mathbf{X}) = \frac{\frac{\partial}{\partial i}\left[\frac{\partial}{\partial i+j} M_{\mathbf{X}}(\mathbf{0})\right] - E(X_1)^2}{Var(X_1)}. \qquad (1)$$

The method proposed by Blacknell consists of obtaining $\mathbf{X}$ as $\sum_{r=1}^{R} H_r^t \mathbf{Y}_r$, with $R \geq 1$ finite, $H_r$ being $N \times N$ matrices and $\mathbf{Y}_1, \ldots, \mathbf{Y}_R$ independent random vectors, each one formed by independent identically distributed random variables obeying Gamma distributions such that $M_{\mathbf{X}}$ has the required properties.

Now notice that if $\mathbf{Y}$ is an $N$-dimensional random vector, $H$ is an invertible matrix and $\mathbf{X} = H^t\mathbf{Y}$, then $\mathbf{X}$ has its moment generating function given by $M_{\mathbf{X}}(\mathbf{s}) = M_{\mathbf{Y}}(H\mathbf{s})$. Also if $\mathbf{Y}_1, \ldots, \mathbf{Y}_R$ are independent random vectors, $H_1, \ldots, H_R$ are $N \times N$ matrices, and $\mathbf{X} = \sum_{r=1}^{R} H_r^t Y_r$, then $M_{\mathbf{X}}(\mathbf{s}) = \prod_{r=1}^{R} M_{\mathbf{Y}_r}(H_r\mathbf{s})$.

Given $L$ such that $1 \leq L \leq N$ define $\mathcal{V}_L = \{\ell = (\ell_1, \ldots, \ell_N) : \ell_1 = 1, \ell_i \in \{0, 1\}, \sum_{i=1}^{N} \ell_i = L\}$; then for each $\ell \in \mathcal{V}_L$, the circulant $N \times N$ matrix $H_{\ell,L}$ is defined as

$$H_{\ell,L} = \frac{1}{L}\begin{bmatrix} \ell_1 & \ell_2 & \ldots & \ell_N \\ \ell_N & \ell_1 & \ldots & \ell_{N-1} \\ \vdots & \vdots & \ldots & \vdots \\ \ell_2 & \ell_3 & \ldots & \ell_1 \end{bmatrix}.$$

These matrices have the property that rows and columns have $L$ non zero values.

Consider $\mathbf{Y} = (Y_1, \ldots, Y_N)^t$ a vector of uncorrelated $Y_i \sim \Gamma(\frac{a\alpha}{L}, \frac{\beta}{L})$ distributed random variables, then for every $\mathbf{t}$ such that $|\mathbf{t}| < \frac{\beta}{L}$, $M_{\mathbf{Y}}(\mathbf{t}) = \prod_{i=1}^{N}(1 - \frac{1}{\beta}Lt_i)^{-\frac{a\alpha}{L}}$. Therefore, if $\mathbf{X} = H_{\ell,L}\mathbf{Y}$, one has that

$$\begin{aligned} M_{\mathbf{X}}((s_1, \ldots, s_N)^t) &= M_{\mathbf{Y}}(H_{\ell,L}\mathbf{s}) \\ &= \prod_{i=1}^{N}(1 - \frac{1}{\beta}L(\sum_{j=1}^{N} h_{i,j}s_j))^{-\frac{a\alpha}{L}}. \end{aligned}$$

Therefore, for each $1 \leq k \leq N$,

$$M_{\mathbf{X}}((0, \ldots, s_k, \ldots, 0)^t) = (1 - \frac{1}{\beta}s_k)^{-\alpha a}.$$

because there are only $L$ rows where $h_{i,k} \neq 0$ and, thus, $\mathbf{X}$ obeys the correlated Gamma distribution with $X_i \sim \Gamma(a\alpha, \beta)$. The coefficients of correlation can be evaluated using eq. (1), or from the moment generating function at the desired lag $j_0$:

$$\begin{aligned} &M_{\mathbf{X}}(s_{k_0}, s_{k_0+j_0}) = \\ &= \prod_{i=1}^{N}(1 - L\frac{1}{\beta}(h_{ik_0}s_{k_0} + h_{i(k_0+j_0)}s_{k_0+j_0}))^{-\frac{a\alpha}{L}}, \end{aligned}$$

and comparing this function with the bivariate case, since

$$\begin{aligned} &M_{X_1,X_2}(s_1, s_2) = \\ &\quad [(1 - \frac{1}{\beta}s_1)(1 - \frac{1}{\beta}s_2)]^{-\alpha(1-\rho)}(1 - \frac{1}{\beta}(s_1 + s_2))^{-\alpha\rho}. \end{aligned}$$

From this it is immediate the identification of the coefficient $\rho$.

In both cases the obtainment of $\rho_j$ as a function of $a$ and $L$ is complicated, and only available in particular cases. It is also noteworthy that it is not possible to establish a specific correlation with a single free parameter, so additional parameters are required.

Finally, the algorithm for the generation of the vector $\mathbf{X}$ with correlated Gamma distribution can be posed as:

1. Define $\rho_1, \ldots, \rho_R$ the desired correlation coefficients for the first $R$ lags, $\alpha$ and $\beta$ the shape and scale parameters for the final marginal distributions, and $N$ the dimension of the final vector.

2. Define $L_1, \ldots, L_m$ integers with $1 \leq L_i \leq N$ and for each of them let $\ell_i \in \mathcal{V}_{L_i}$ be such that they generate filters $H_{\ell_1}, \ldots, H_{\ell_m}$. These filters induce non-null autocorrelation functions only in the first $R$ lags.

3. Calculate $a_1, \ldots, a_m$ such that

$$\begin{bmatrix} 1 \\ \rho_1 \\ \vdots \\ \rho_R \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ \rho_1(H_{\ell_1} Y_1) \\ \vdots \\ \rho_R(H_{\ell_1} Y_1) \end{bmatrix} + \cdots + a_m \begin{bmatrix} 1 \\ \rho_1(H_{\ell_m} Y_m) \\ \vdots \\ \rho_R(H_{\ell_m} Y_m) \end{bmatrix},$$

   where $\mathbf{Y}_1, \ldots, \mathbf{Y}_m$ are $N$-dimensional independent vectors that marginally obey $Y_{i,j} \sim \Gamma(\frac{a_i \alpha}{L}, \frac{\beta}{L})$ distributions , for every $1 \leq i \leq R$ and every $1 \leq j \leq N$.

4. Return $\mathbf{X} = \sum_{i=1}^{m} H_{\ell_i, L_i} \mathbf{Y}_i$.

   Note that

$$M_{\mathbf{X}}(\mathbf{s}) = \prod_{r=1}^{m} \prod_{i=1}^{N} (1 - \frac{L}{\beta} \sum_{j=1}^{N} h_{r,i,j} s_j)^{-\frac{a_r \alpha}{L}},$$

then $M_{\mathbf{X}}((0, \ldots, s_k, \ldots, 0)^t) = (1 - \frac{1}{\beta} s_k)^{-\alpha}$ and $\mathbf{X}$ has marginal Gamma distributions with the desired parameters $\alpha$ and $\beta$.

This algorithm is relatively simple, though computationally more and more expensive as the number of non-null correlated random variables increases. This limits its usefulness to "small" cases, where the biggest non-null correlation lags are of order 2 or 3 at the most.

### 2.3 Transformation Method

An alternative approach to the problem of generating outcomes from correlated Gamma vectors is a method based in three steps:

1. generating independent outcomes from a convenient distribution;

2. introducing correlation in these data;

3. transforming the correlated observations into the desired marginal properties [10].

The transformation that guarantees this is obtained from the cumulative distribution functions of the data obtained in step 2 and that of the desired distributions. The reader is invited to recall that if $U$ is a continuous random variable with cumulative distribution function $F_U$ then $F_U(U)$ obeys a $\mathcal{U}(0,1)$ law and, reciprocally, if $V$ obeys a $\mathcal{U}(0,1)$ distribution then $F_U^{-1}(V)$ is $F_U$ distributed. In order to use this method it is neccesary to know the correlation of the random variables after the transformation.

In principle, there are no restrictions on the possible order parameters values that can be obtained by this method, but numerical issues must be taken into account. Other important point is that not every desired final correlation structure is mapped onto a feasible intermediate correlation structure.

Consider any $\alpha > 0$ and let $G$ be the cumulative distribution function of a $\Gamma(\alpha, \alpha)$ distributed random variable

$$G(y) = \frac{(\alpha)^\alpha}{\Gamma(\alpha)} \int_0^y x^{\alpha-1} e^{-\alpha x} dx.$$

Let now $\Phi$ be the cumulative distribution function of a standard Gaussian random variable (denote this distribution $\mathcal{N}(0,1)$). Since $U \sim \mathcal{N}(0,1)$ then the variable $G^{-1}(\Phi(U)) = X \sim \Gamma(\alpha, \alpha)$.

Consider now the $N^2$-dimensional random vector $(U_1, \ldots, U_{N^2})$ with $\mathcal{N}(\mathbf{0}, \Sigma)$ distribution, where

$$\Sigma = \begin{bmatrix} 1 & \rho_{1,2} & \ldots & \rho_{1,N^2} \\ \rho_{1,2} & 1 & \ldots & \rho_{2,N^2} \\ \vdots & \ddots & \ldots & \vdots \\ \rho_{1,N^2} & \rho_{2,N^2} & \ldots & 1 \end{bmatrix} \tag{2}$$

with $0 \leq |\rho_{i,j}| < 1$, for every $1 \leq i \leq N^2 - 1$ and every $i + 1 \leq j \leq N^2$. Define for every $1 \leq k \leq N^2$ the random variable $X_k = G^{-1}(\Phi(U_k))$; then $\mathbf{X} = (X_1, \ldots, X_{N^2})^t$ has a correlated Gamma distribution with $\rho(X_k, X_l) = \alpha(E(X_k . X_l) - 1)$. Now

$$E(X_k X_l) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^{-1}(\Phi(u_k)) G^{-1}(\Phi(u_l)) \phi_2 du_k du_l,$$

where

$$\phi_2 = \frac{1}{2\pi\sqrt{(1 - \rho_{k,l}^2)}} \exp\left( -\frac{u_k^2 - 2\rho_{k,l} u_k u_l + u_l^2}{2(1 - \rho_{k,l}^2)} \right).$$

Since the function $G^{-1}$ is only available using numerical methods, it is an approximation that may impose restrictions to the use of this simulation method.

### 2.4 The sum of squared normals

It is known that the sum of the squares of $n$ independent identically standard Gaussian random variables

obeys a Gamma distribution with shape parameter $n/2$. If another Gaussian vector is generated, with the same distribution and independence structure, but with correlation between corresponding coordinates in the first vector, then the Gamma random variable obtained from the second vector will be correlated with that obtained from the first one.

This procedure, described for the bivariate case in [2] and easily generalised to any finite number of Gamma random variables, has the disadvantage of only allowing shape parameters taking values $n/2$ with $n$ integer. The correlation between components is required to be the square root of the final desired correlation, which constitutes another restriction of the method.

A proof of the properties of correlated Gamma fields obtained by this method is presented, and in the following section this scheme will be extended to allow the use of convolution in Gaussian vectors.

**Proposition 2.1** *Consider the $\alpha$ independent random vectors $\underline{\xi}_1, \ldots, \underline{\xi}_\alpha$, each of dimension $N^2$, all obeying the $N(\mathbf{0}, \Sigma)$ distribution, such that $\Sigma$ is of the form*

$$\Sigma = \frac{1}{2}\Sigma_1 \tag{3}$$

*with $\Sigma_1$ given in eq. (2), $0 \leq \rho_{i,j} < 1$ for every $1 \leq i \leq N^2 - 1$ and $i+1 \leq j \leq N^2$.*

*Consider $\underline{\eta} = \sum_{j=1}^{\alpha} \underline{\xi}_j^2$, then $\underline{\eta} = (\eta_1, \ldots, \eta_{N^2})^t$ has the correlated Gamma distribution such that $\eta_i \sim \Gamma(\alpha/2, 1)$ and the correlation between $\eta_i$ and $\eta_j$ is $\rho_{i,j}^2$, for every $1 \leq i \leq N^2 - 1$ and $i+1 \leq j \leq N^2$.*

*Also if $1/\beta_1, \ldots, 1/\beta_{N^2}$ are positive integers and if $B$ is the diagonal matrix formed by these constants, then $X' = B\underline{\eta}$ has correlated Gamma distribution with marginals $\Gamma(\alpha/2, \beta_k)$ and with correlation between $X_i'$ and $X_j'$ given by $\rho_{i,j}^2$, for every $1 \leq i \leq N^2 - 1$ and $i+1 \leq j \leq N^2$.*

**Proof:** Some useful and well known results are:

1. If $\xi \sim N(0, 1/2)$, then $\xi^2 \sim \Gamma(1/2, 1)$.

2. Consider the independent identically distributed random variables $\xi_1, \ldots, \xi_\alpha$ obeying the $N(0, 1/2)$ law, then $\xi_1^2 + \cdots + \xi_\alpha^2 \sim \Gamma(\alpha/2, 1)$.

3. Consider $\underline{\xi} = (\xi_1, \ldots, \xi_{N^2})^t$ an $N^2$-dimensional vector with $N(\mathbf{0}, \Sigma)$ distribution, where $\Sigma$ is as given in eq. (3). Let $\underline{\xi}_j = (\xi_{1,j}, \ldots, \xi_{N^2,j})^t$ with $1 \leq j \leq \alpha$, be $N^2$-dimensional independent random vectors, each having a $N(\mathbf{0}, \Sigma)$ distribution, with $\Sigma$ of the form given in eq. (3). Define $\underline{\eta} = \sum_{j=1}^{\alpha} \underline{\xi}_j^2 = (\sum_{j=1}^{\alpha} \xi_{1,j}^2, \ldots, \sum_{j=1}^{\alpha} \xi_{N^2,j}^2)^t$ and let $\eta_i = \sum_{j=1}^{\alpha} \xi_{i,j}^2$, with $1 \leq i \leq N^2$. Then $\eta_i \sim \Gamma(\alpha/2, 1)$, with $E(\eta_j) = \alpha/2$ and $Var(\eta_j) = \alpha/2$. In other words, $\underline{\eta}$ has a correlated Gamma distribution.

In order to compute the correlation between $\eta_i$ and $\eta_j$, let us verify first that if $(U, V)$ is a $N((0,0), \Sigma)$ distributed vector with covariance matrix of the form

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

then $E(U^2V^2) = \sigma_1^2\sigma_2^2(1 + 2\rho^2)$. Using this, since $Cov(\eta_i, \eta_j) = E(\eta_i\eta_j) - E(\eta_i)E(\eta_j)$, and since $E(\eta_j) = E(\eta_i) = \alpha/2$, we must compute $E(\eta_i, \eta_j)$. Using the fact that the vectors $\underline{\xi}_j$ are independent, the previous result and the fact that $E(\xi_{i,k}^2) = 1/2$, we have

$$\begin{aligned}
E(\eta_i, \eta_j) &= E(\sum_{h=1}^{\alpha} \xi_{i,h}^2 \sum_{k=1}^{\alpha} \xi_{j,k}^2) = \\
&= \sum_{h=1}^{\alpha}\sum_{k=1}^{\alpha} E(\xi_{i,h}^2 \xi_{j,k}^2) \\
&= \sum_{h=1}^{\alpha} E(\xi_{i,h}^2 \xi_{j,h}^2) + \sum_{h=1}^{\alpha}\sum_{k \neq h} E(\xi_{i,k}^2)E(\xi_{j,k}^2) \\
&= \alpha E(\xi_{i,i}^2 \xi_{j,i}^2) + \alpha(\alpha - 1)E(\xi_{1,1}^2)^2 \\
&= \frac{\alpha}{4}(1 + 2\rho_{i,j}^2) + \alpha(\alpha - 1)\frac{1}{4} \\
&= \frac{\alpha}{4}(\alpha + 2\rho_{i,j}^2)
\end{aligned}$$

Then $Cov(\eta_i, \eta_j) = \frac{1}{4}\alpha^2 + \frac{\alpha}{2}\rho_{i,j}^2 - \frac{\alpha^2}{4} = \frac{\alpha}{2}\rho_{i,j}^2$, so

$$\rho(\eta_i, \eta_j) = \frac{Cov(\eta_i, \eta_j)}{\sqrt{Var(\eta_i)Var(\eta_j)}} = \frac{\frac{\alpha}{2}\rho_{i,j}^2}{\frac{\alpha}{2}} = \rho_{i,j}^2.$$

Summarizing, consider $\underline{\eta} = (\eta_1, \ldots, \eta_{N^2})^t$ is a $N^2$-dimensional vector with $\eta_j \sim \Gamma(\alpha/2, 1)$ and $\rho(\eta_i, \eta_j) = \rho_{i,j}^2$. If $B = \text{Diagonal}(\frac{1}{\beta_1}, \ldots, \frac{1}{\beta_{N^2}})$ with $\beta_i > 0$ and $X' = B\underline{\eta}$ then $\eta_i \sim \Gamma(\alpha/2, \beta_i)$ and $Cov(X_i', X_j') = Cov(\eta_i, \eta_j) = \rho_{i,j}^2$.

### 2.5 Proposal: multivariate reduction

The last method has a restriction on the possible values for the shape parameter, but it has the advantage of being easy to implement and the restriction is of no practical importance for the applications that we bear in mind. This method relies on the obtainment of correlated Gamma random variables, with correlations that are the square root of the desired value.

The use of convolution filters for the generation of such correlated Gamma deviates is proposed in this work, using independent normal random variables as input. The procedure can be outlined as

1. Generate independent normal observations.
2. Choose the correlation as the square of a suitable function $E$, defined on $\mathbf{Z}^2$.
3. Calculate the mask $\theta$ that the convolution filter will use, such that $\theta * \theta = E$.
4. Apply the convolution filter to the independent normal deviates, obtaining outcomes from the processes with correlation $E$ in each component.
5. Return the sum of the squares of each normal deviate.

We will consider the family of functions $E$ such that

1. $E : \mathbf{Z}^2 \to \mathbf{R}$ is a periodic function with fundamental period $R_N = \{(s_1, s_2)/0 \leq s_1, s_2 \leq N-1\}$.
2. There is a uni-dimensional periodic function $E_1$ such that $E(s_1, s_2) = E_1(s_1)E_1(s_2)$.
3. There is a real characteristic function $c$ such that

$$E_1(s) = \begin{cases} c(s) & 0 \leq s \leq \frac{N}{2} \\ c(N-s) & \frac{N}{2}+1 \leq s \leq N-1 \end{cases}$$

The use of this technique will be illustrated with a particular (useful and widely employed) characteristic function: that of normal distribution. The following definitions are needed:

**Definition 2.1** *Denote* $\mathrm{s} = (s_1, s_2)$ *and let* $E : \mathbf{Z}^2 \to \mathbf{R}$ *be the periodic function with fundamental period* $R_N$ *defined by*

$$E(\mathrm{s}) = \begin{cases} \exp(-\frac{1}{2}\frac{s_1^2+s_2^2}{\ell^2})) & \text{if } \mathrm{s} \in R_1 \\ \exp(-\frac{1}{2}\frac{(N-s_1)^2+s_2^2}{\ell^2}) & \text{if } \mathrm{s} \in R_2 \\ \exp(-\frac{1}{2}\frac{s_1^2+(N-s_2)^2}{\ell^2}) & \text{if } \mathrm{s} \in R_3 \\ \exp(-\frac{1}{2}\frac{(N-s_1)^2+(N-s_2)^2}{\ell^2}) & \text{if } \mathrm{s} \in R_4 \end{cases}$$

*where* $R_1 = \{\mathrm{s} : 0 \leq s_1, s_2 \leq N/2\}$, $R_2 = \{\mathrm{s} : N/2+1 \leq s_1 \leq N-1, 0 \leq s_2 \leq N/2\}$, $R_3 = \{\mathrm{s} : 0 \leq s_1 \leq N/2, N/2+1 \leq s_2 \leq N-1\}$, *and* $R_4 = \{\mathrm{s} : N/2+1 \leq s_1, s_2 \leq N-1\}$.

**Definition 2.2** *Let* $\theta : \mathbf{Z}^2 \to \mathbf{R}$ *be a periodic function with fundamental period* $R_N$ *such that*

$$\begin{aligned} \theta * \theta(s_1, s_2) &= \sum_{t_1=0}^{N-1}\sum_{t_2=0}^{N-1} \theta(t_1, t_2)\theta(s_1-t_1, s_2-t_2) \\ &= E(s_1, s_2) \end{aligned}$$

*and such that*

$$\theta(s_1, s_2) = \begin{cases} \theta(N-s_1, s_2) & \text{if } (s_1, s_2) \in R_2 \\ \theta(s_1, N-s_2) & \text{if } (s_1, s_2) \in R_3 \\ \theta(N-s_1, N-s_2) & \text{if } (s_1, s_2) \in R_4 \end{cases}$$

**Proposition 2.2** *There is a function* $\theta : \mathbf{Z}^2 \to \mathbf{R}$ *that satisfies the previous definition.*

**Proof:** The function $E$ belongs to the family since $E(s_1, s_2) = \exp(-\frac{1}{2}\frac{s_1^2}{\ell^2})\exp(-\frac{1}{2}\frac{s_2^2}{\ell^2}) = E_1(s_1)E_1(s_2)$.

We will prove that there exists $\theta_1 : \mathbf{Z} \to \mathbf{R}$ unidimensional with period $R = \{0, \ldots, N-1\}$ such that

1. $\theta_1 * \theta_1(s_1) = \sum_{j=0}^{N-1} \theta_1(j)\theta_1(s_1-j) = E_1(s_1)$ for every $s_1 \in \mathbf{Z}$,
2. $\theta_1(s_1) = \theta_1(N-s_1)$, if $\frac{N}{2}+1 \leq s_1 \leq N-1$.

If such $\theta_1$ exists, it will suffice to define the function $\theta$ in separable form, i.e., $\theta(s_1, s_2) = \theta_1(s_1)\theta_1(s_2)$ in order to hold the proposition.

Using lemma 2.1, the Fourier transform of $E_1$

$$\hat{E_1}(s_1) = \frac{1}{N}\sum_{k=0}^{N-1} E_1(k)\omega^*_{s_1 k, N}$$

is a real positive function, then we can define the periodic function $\psi$ as $\psi = \sqrt{\hat{E_1}}$. This function satisfies that $\psi(k) = \psi(N-k)$, since

$$\begin{aligned} \hat{E_1}(N-k) &= \frac{1}{N}\sum_{l=0}^{N-1} E_1(l)\omega^*_{(N-k)l, N} \\ &= \frac{1}{N}\sum_{l=0}^{N-1} E_1(N-l)\omega^*_{(N-k)l, N} \\ &= \hat{E_1}(k). \end{aligned}$$

In order to obtain this result the properties of the unit roots and the definition of $E_1$ are used.

Consider now $\theta_1$, the inverse Fourier transform of $\psi$

$$\theta_1(s_1) = \tilde{\psi}(s_1) = \sum_{k=0}^{N-1} \psi(k)\omega_{k s_1, N}.$$

Then, by the properties of the periodic Fourier transform and the definition of $\theta_1$, $\widehat{\theta_1 * \theta_1} = \hat{\theta_1}\hat{\theta_1} = \hat{\tilde{\psi}}.\hat{\tilde{\psi}} = \psi.\psi = \hat{E_1}$, and by the unicity of the transform one has that $\theta_1 * \theta_1 = E_1$ verifying, thus, the first condition.

The second condition stems from the fact that the inverse Fourier transform always satisfies that $\tilde{\psi}(N-k) = \tilde{\psi}(k)^*$ for every $0 \leq k \leq N-1$, and that

$$\tilde{\psi}(k)^* = \sum_{l=0}^{N-1} \psi(l)\omega^*_{kl, N}$$

$$= \sum_{l=0}^{N-1} \psi(N-l)\omega_{(N-l)k,N} = \tilde{\psi}(k).$$

From these one has that $\theta_1(s_1) = \theta_1(N-s_1)$ if $N/2+1 \leq s_1 \leq N-1$.

**Lemma 2.1** *The Fourier transform of $E_1$, given by $\hat{E}_1(s_1) = \frac{1}{N}\sum_{k=0}^{N-1} E_1(k)\omega^*_{s_1 k,N}$ is a real positive function.*

**Proof:** We will now check that $\hat{E}_1$ is a real positive function. Remember that $E_1(j) = c(j)$, for every $0 \leq j \leq N/2$ and $E_1(N-j) = c(N-j)$ in every $1 \leq j \leq N/2-1$, with $c$ the characteristic function of the $\mathcal{N}(0,\ell^{-2})$ distribution. Since $c$ is a positive definite function, then

$$M = \begin{bmatrix} E_1(0) & E_1(1) & \dots & E_1(N-1) \\ E_1(N-1) & E_1(0) & \dots & E_1(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ E_1(1) & E_1(2) & \dots & E_1(0) \end{bmatrix}$$

is a circulant positive definite matrix. Therefore, its eigenvalues are positive real numbers. These eigenvalues are, for every $0 \leq j \leq N-1$,

$$\lambda_j = \sum_{k=0}^{N-1} E_1(k)\omega^*_{jk,N} = N\hat{E}_1(j) > 0,$$

and, therefore, $\hat{E}_1$ is a real positive function.

**Definition 2.3** *Consider $\zeta_k$, $1 \leq k \leq 2\alpha$, independent Gaussian white noise periodic stochastic processes with fundamental period $R_N$. Define $\xi_k$, $1 \leq k \leq 2\alpha$, periodic processes with fundamental period $R_N$, as*

$$\xi_k(s_1,s_2) = (\theta * \zeta_k)(s_1,s_2)$$

**Proposition 2.3** *The processes $\xi_k$ as previously defined satisfy the following properties:*

1. *$\xi_k(s_1,s_2) \sim N(0,(\theta*\theta)(0,0)/2)$, i.e., $\xi_k$ are stochastic processes with Gaussian marginals with zero mean and variances $1/2$.*
2. *$E(\xi_k(0,0)\xi_k(s_1,s_2)) = \frac{(\theta*\theta)(s_1,s_2)}{2} = \frac{E(s_1,s_2)}{2}$.*
3. *$\rho(\xi_k(0,0),\xi_k(s_1,s_2)) = E(s_1,s_2)$*

**Proof:** Since the periodic convolution is a finite linear combination, the processes $\xi_k$ obey Gaussian distributions since the processes $\zeta_k$ are independent white noise Gaussian processes. In order to verify the second item, the same reason is used along with the definition of $\theta$.

$$\begin{aligned} E(\xi_k(0,0)\xi_k(s_1,s_2)) &= E(\sum_{t,n} \zeta_k(t_1,t_2)\cdot \\ &\quad \theta(s_1-t_1,s_2-t_2)\zeta_k(n_1,n_2)\theta(-n_1,-n_2)) \\ &= \sum_{t,n} \theta(-n_1,-n_2)\theta(s_1-t_1,s_2-t_2)\cdot \\ &\quad E(\zeta_k(t_1,t_2)\zeta_k(n_1,n_2)) \\ &= \sum_{n} \theta(-n_1,-n_2)\theta(s_1-n_1,s_2-n_2)\cdot \\ &\quad E(\zeta_k(n_1,n_2)\zeta_k(n_1,n_2)) \\ &= \frac{1}{2}\sum_{n} \theta(n_1,n_2)\theta(s_1-n_1,s_2-n_2) \\ &= \frac{1}{2}(\theta*\theta)(s_1,s_2) = \frac{1}{2}E(s_1,s_2) \end{aligned}$$

Also note that $\rho(\xi(0,0),\xi(s_1,s_2)) = E(s_1,s_2)$.

**Definition 2.4** *Define the periodic stochastic process $\eta$ with fundamental period $R_N$ as the sum of squares $\eta(s_1,s_2) = \sum_{k=1}^{2\alpha}\xi_k^2(s_1,s_2)$ for every $(s_1,s_2) \in R_N$, and assume $\beta > 0$. The periodic stochastic process $\sigma$ is defined as $\sigma(s_1,s_2) = \frac{1}{\beta}\eta(s_1,s_2)$ for every $(s_1,s_2) \in R_N$.*

**Proposition 2.4** *The following properties hold*

1. *The process $\eta$ is a weakly stationary stochastic process with correlated Gamma distribution such that $\eta(s_1,s_2) \sim \Gamma(\alpha,1)$.*
2. *The process $\sigma$ is a weakly stationary stochastic process with correlated Gamma distribution such that*
   (a) *$\sigma(s_1,s_2) \sim \Gamma(\alpha,\beta)$, then $E(\sigma(s_1,s_2) = \frac{\alpha}{\beta}$ and $Var(\sigma(s_1,s_2) = \frac{\alpha}{\beta^2}$.*
   (b) *The coefficient of correlation at lag $(s_1,s_2)$ is $\rho(\sigma_{(s_1,s_2)},\sigma_{(0,0)}) = E^2(s_1,s_2)$.*

**Proof:** Note that the processes $\xi_1,\dots,\xi_{2\alpha}$ are independent weakly stationary Gaussian processes, each with

1. $\xi_k(s_1,s_2) \sim N(0,1/2), \forall\,(s_1,s_2) \in R_N$ and
2. $E(\xi_k(s_1,s_2)\xi_k(t_1,t_2) = \frac{1}{2}E(s_1-t_1,s_2-t_2)$.

Applying Proposition 2.1 to $\xi_k$ in $R_N$, we obtain the correlated Gaussian process $\eta$ with marginal distributions $\eta(s_1,s_2) \sim \Gamma(\alpha,1)$. Analogously, $\sigma$ is a periodic process with correlated Gamma distribution and

$\sigma \sim \Gamma(\alpha, \beta)$, with coefficients of correlation given by

$$\begin{aligned} \rho(\sigma_{s_1,s_2}, \sigma_{(0,0)}) &= \rho(\eta(s_1,s_2), \eta(0,0)) \\ &= \rho^2(\xi(s_1,s_2), \xi(0,0)) \\ &= E^2(s_1,s_2). \end{aligned}$$

## 3 Simulating heterogeneous images

The previously presented method was implemented using the IDL5.2 development platform, and as the following algorithm

1. Generate the Gaussian white noises $\zeta_k$, with variance 1/2 for every $1 \leq k \leq 2\alpha$.
2. Define $e_1(j) = \exp(-\frac{1}{2}\frac{j^2}{\ell^2})$ if $0 \leq j \leq N/2$, and $e_1(j) = e_1(N-j)$ if $N/2+1 \leq j \leq N-1$.
3. Compute the frequency domain mask $\psi_2(s_1,s_2) = \sqrt{\mathrm{FFT}(e_1,-1)(s_1)}.\sqrt{\mathrm{FFT}(e_1,-1)(s_2)}$.
4. Calculate $\xi_k = \mathrm{FFT}(\psi_2.\mathrm{FFT}(\zeta_k,-1),1)$, for every $1 \leq k \leq 2\alpha$.
5. Obtain $\sigma = \frac{1}{\beta}\sum_{k=1}^{2\alpha} \xi_k^2$.
6. Generate independent random variables identically distributed as $\Gamma(n,n)$, where $n$ is the desired equivalent number of looks, $Y$.
7. Return $Z = \sigma.Y$.

The notation $\mathrm{FFT}(U,-1)$ and $\mathrm{FFT}(U,1)$ represents the direct and inverse Fourier transforms, respectively, of the input $U$. IDL 5.2 computes these functions using a routine based on the Fast Fourier Transform algorithm. It is noteworthy that the bigger the parameter $\alpha$, the slower will be the execution of this procedure.

The return simulated with this procedure obeys an intensity K distribution, characterised by the density

$$f_Z(z) = \frac{2\left(\sqrt{\lambda n}\right)^{\alpha+n} z^{(\alpha+n)/2} K_{\alpha-n}\left(2\sqrt{\lambda n z}\right)}{\Gamma(n)\,\Gamma(\alpha)}$$

where $z, \alpha, \lambda, n > 0$ and $K_v$ is the modified Bessel function of the third kind and order $\nu$. This is the distribution of a random variable obtained as the product of two independent random variables that obey $\Gamma(n,n)$ and $\Gamma(\alpha,\lambda)$ distributions. This distribution has been consagrated in the SAR literature as an excellent model for heterogeneous and homogeneous targets. More details about this and other distributions arising from the multiplicative model can be seen in [4].

Fig. 1 shows sixteen simulated $\Gamma$ fields of size $256 \times 256$ each, with varying shape parameter $\alpha$ (columns with $\alpha \in \{0.5, 1, 1.5, 2\}$) and correlation length $\ell$ (rows with $\ell \in \{1, 2, 4, 8\}$). Fig. 2 shows the images that should be returned by a three-looks system, corresponding to the truth images shown in Fig. 1.

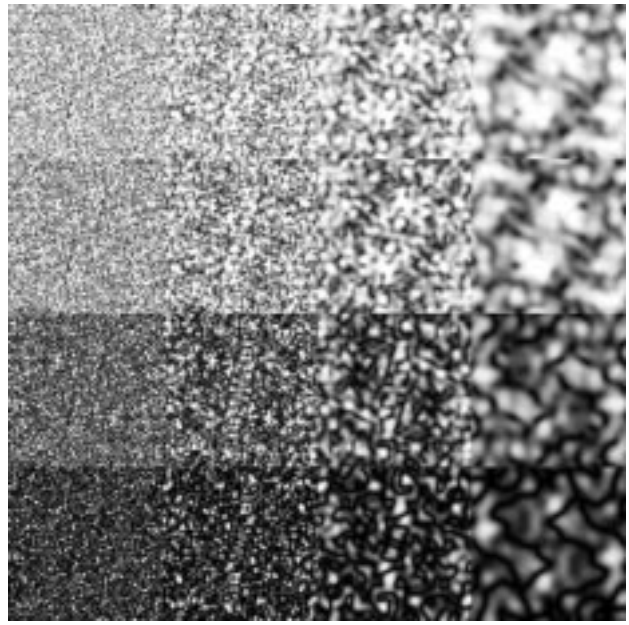

Figure 1: Simulated $\sigma$ fields, with $\alpha$ varying in the rows and correlation lag varying in the columns.

The adequacy of the simulation procedure was checked comparing desired correlation structures with the observed ones, for a variety of parameters, and the results are compatible with the theory. Detailed results on estimation procedures for the spatial dependence of SAR data will be reported elsewhere.

## 4 Conclusions

Methods for the generation of correlated Gamma fields have been presented and discussed, aiming at the simulation of correlated $\mathcal{K}$ fields for SAR image simulation. Moving averages has the advantage of allowing any shape parameter and a wide variety of autocorrelation functions but, in practice, it is too cumbersome to be implemented but in very simple situations. Methods based on random variables transformations are also very general and have the least restrictions of all the techniques considered, but they rely on numerical approximations which are seldom effective. The sum of squares of Gaussian random variables limits the shape parameters to halves of integers, but if this restriction is of little or no significance (as is the case for SAR image simulation) it is the recommended method.

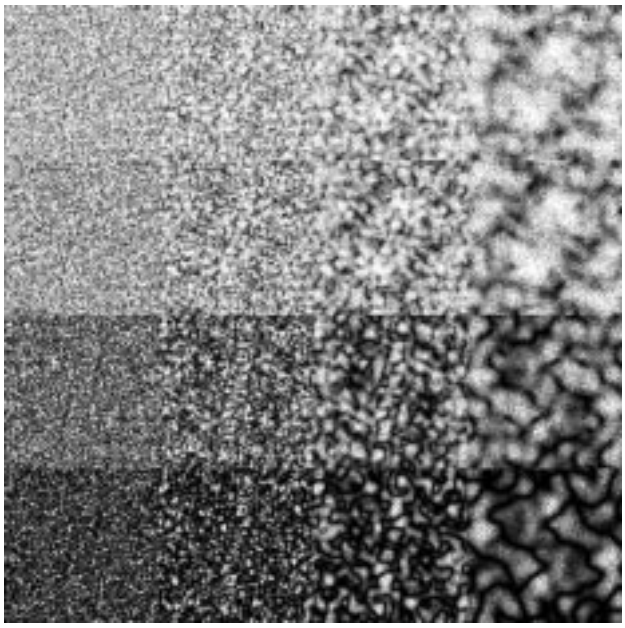

Figure 2: The observed data, with three looks and ground truth simulated in Fig. 1.

A family of autocorrelation functions was proposed in this article, and a simulation methodology was presented for it. Members of this family have been previously used for the modelling of forest data [9], assuming that the spatial correlation decays exponentially at distance $\ell$. Simulations of this model are presented for several parameter values.

This work will be extended to other members of the class of distributions arising in the modelling and analysis of SAR images.


### Acknowledgements

This work was partially supported by grants from the Argentine agencies CONICOR and Secyt-UNC, and the Brazilian agencies CNPq (Proc. 523469/96-9) and FACEPE (APQ 0707-1.03/97). The authors acknowledge the suggestions made by the referres.


### References


[1] B.C. Arnold (1967). A note on multivariate distributions with specified marginals. *J. Am. Stat. Assoc.*, v. 62, p. 1460–1461.

[2] B.B. Armstrong, H.D. Griffiths (1991) Modelling spatially correlated $\mathcal{K}$- distributed clutter, *Elect. Let.*, v. 27, p. 1355–1356.

[3] D. Blacknell (1994) New method for the simulation of correlated K-distributed clutter. *IEE Proc. Rad., Son. Nav.*, v. 141, p. 53–58.

[4] A.C. Frery, H.-J. Müller, C.C.F. Yanasse, S.J.S. Sant'Anna (1997) A model for extremely heterogeneous clutter, *IEEE Transactions on Geoscience and Remote Sensing*, v. 35, p. 648–659.

[5] N.L. Johnson, S. Kotz (1972) *Distributions in statistics: Continuous Multivariate Distributions.* Wiley, New York.

[6] P.A.W. Lewis (1983) Generating negatively correlated Gamma variates using the Beta-Gamma transformation, *Proc. 1983 Winter Simulation Conf.*, Washington, D.C., p. 175–176.

[7] P.A.W. Lewis (1985) Some simple models for continuous variate time series. *Water Resources Bull.*, v. 21, p. 635–644.

[8] P.A.W. Lewis, E.J. Orav (1989). *Simulation methodology for statisticians, operations analysts, and engineers.* Wadswoth, Brooks and Cole. Statistics/Probability Series.

[9] P. Lombardo, C.J. Oliver (1995) Estimating the correlation properties of $\mathcal{K}$-distributed SAR clutter. *IEE Proc. Rad. Son. Nav.*, v. 142, p. 167–178.

[10] C.J. Oliver; S. Quegan (1998) *Understanding synthetic aperture radar images.* Artech House, Boston.

[11] G. Ronning (1977) A simple scheme for generating Multivariate Gamma distributions with non negative covariance matrix, *Technometrics*, v. 19, p. 179–183.

[12] B.W. Schmeiser, R. Lal (1982), Bivariate Gamma random vectors, *Oper. Res.*, v. 30, p. 266–274.