A scaled morphological toggle operator for image transformations

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Abstract

Scale dependent signal representations have proved to be useful in several image processing applications. In this paper, we define a toggle operator for binarization/segmentation purposes based on scaled versions of an image transformed by morphological operations. The toggle decision rule, determining the new value of a pixel, considers local spatial information, in contrast to other multiscale approaches that takes into account mainly global information (e.g., the scale signal under study). We show that the proposed operator can identify significant image extrema information in such a way that when it is used in a binarization process yields very good segmentation and filtering results. Our algorithm is validated against known threshold-based segmentation methods using images of different classes and subjected to different lighting conditions.

1. Introduction

Different representation levels to extract significant features of a signal have been useful in several image and signal processing applications. Approaches based on wavelets, pyramidal decomposition and scale-space theory are largely used, the latter one being the focus of this work.

Scale-space is a mathematical concept from which it is possible to relate information obtained in different scales, an intrinsic and general problem of the multiscale approaches. In the scale-space theory [23, 3], the representation of an interest feature of the signal describes a continuous path through the different scales. In other words, if important features of the signal are present at scale n, then are also present in all the scale-space path up to the original image representation (at scale $\sigma = 0$). This property is called monotonicity, since the number of features must necessarily be a monotonic decreasing function of the scale σ [23], having no creation of features.

Scaled morphological operators have been frequently associated to non-linear filters and scale-space theory. Using openings and closings, Park and Lee [13] defined a scale-space for one-dimensional signals. Jang and Chin [4] also used these operations in the definition of a scale-space where the interest features are the contour segments of binary images. The extension of these results to gray-scale images is not direct.

As proved in [9], any convolution kernel used to obtain the scale-space introduces new extrema as the scale increases and, thus, the monotonicity property for the signal extrema does not hold. To solve this problem, Jackway [3] introduced a scale-space called *Multiscale Morphological Dilation Erosion* (MMDE) that considers non-linear morphological operators [18] in a scaled morphology framework, sharing important properties with the Witkin's scale-space [23].

In the MMDE, a multiscale operation unifies the erosion and dilation transformations so that both positive and negative scales are taken into account. The multiscale erosion/dilation of a signal $f(\mathbf{x})$ by the scale dependent structuring function $g_\sigma(\mathbf{x})$ is defined as [3]:

$$(f \otimes g_\sigma)(\mathbf{x}) = \begin{cases} (f \oplus g_\sigma)(\mathbf{x}), & \text{if } \sigma > 0; \\ f(\mathbf{x}), & \text{if } \sigma = 0; \\ (f \ominus g_\sigma)(\mathbf{x}), & \text{if } \sigma < 0. \end{cases} \quad (1)$$

where $(f \oplus g_\sigma)(\mathbf{x})$ denotes dilation and $(f \ominus g_\sigma)(\mathbf{x})$ erosion of the pixel at location $\mathbf{x}$.

It is easy to see that the image is processed by dilation, for positive scales, and by erosion for the negative ones. Jackway defined the interest features as the watershed of the smoothed signal in a certain scale. However, as stated by the author, this method cannot be directly associated to image segmentation since "the watershed arcs moves spatially with varying scale and are not a subset of those at zero scale" [3, 2].

In [7, 8], Leite and Teixeira explored the important MMDE scale-space property, concerning image extrema preservations, by using the extrema set obtained during the filtering process as markers in a homotopic modification of the original image, thus avoiding the spatial shifting of the watershed lines. They controlled the extrema merging through the different scales, obtaining good segmentation results. The authors also defined a new operator that explores the idempotence of a smoothed signal transformed by the MMDE scale-space, establishing a relation between the structuring function g_σ and the extremes that persist at a given scale σ.

Scaled morphological operators are also applied for image sharpening. Kramer [6] proposed a non-linear operator that replaces the original gray value of a pixel by the local minimum or maximum, depending on what value is closer to the original one. Shavemaker et al. [17] generalized this result by defining a new class of iterative scaled morphological image operators. In fact, they proved that all the operators that use a concave structuring function have interesting sharpening properties.

In this paper, we introduce a new image processing operator based on scaled versions of a signal defined by the scale-space morphological transformations. This operator, defined on the scope of a toggle transformation [20, 21], considers local pixel information (not only scale knowledge) to determine if each pixel should be processed by erosion or dilation. As we will see elsewhere, good segmentation results were obtained through the validation tests carried out on images presenting different lighting conditions, which proves the robustness of our approach.

This paper is organized as follows. In Section 2, we introduce the considered multiscale morphology which constitutes the basis of our operator. Section 3 discusses some basic properties of toggle mapping operators. The proposed operator and its main features are defined in Section 4. In Section 5 we show some results and comparisons with other threshold-based segmentation methods. Finally, some conclusions are drawn in Section 6.

2. The Morphological Scale-space

Mathematical morphology is a non-linear image analysis technique that extracts image object's information by describing its geometrical structures in a formal way [11, 18, 19, 21].

Let $f : \mathcal{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be an image function and $g : \mathcal{G} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a structuring function. The two fundamental operations of gray-scale morphology, erosion and dilation, are defined as:

Definition 2.1 *[21] (Dilation) The dilation of the function $f(\boldsymbol{x})$ by the structuring function $g(\boldsymbol{x})$, $(f \oplus g)(\boldsymbol{x})$, is given by:*

$$(f \oplus g)(\boldsymbol{x}) = \sup_{\boldsymbol{t} \in \mathcal{G} \cap \check{\mathcal{D}}_{-x}} \{f(\boldsymbol{x} - \boldsymbol{t}) + g(\boldsymbol{t})\} \qquad (2)$$

Definition 2.2 *[21] (Erosion) The erosion of the function $f(\boldsymbol{x})$ by the structuring function $g(\boldsymbol{x})$, $(f \ominus g)(\boldsymbol{x})$, is given by:*

$$(f \ominus g)(\boldsymbol{x}) = \inf_{\boldsymbol{t} \in \mathcal{G} \cap \mathcal{D}_{-x}} \{f(\boldsymbol{x} - \boldsymbol{t}) - g(\boldsymbol{t})\} \qquad (3)$$

where $\mathcal{D}_x$ is the translate of $\mathcal{D}$, $\mathcal{D}_x = \{\mathbf{x} + \mathbf{t} : \mathbf{t} \in \mathcal{D}\}$, and $\check{\mathcal{D}}$ is the reflection of $\mathcal{D}$. In the discrete case $\max$ and $\min$ are used [1].

The result of these two operations depends on the location of the structuring function center. To avoid level-shifting and horizontal translation effects, respectively, one must consider that [3]

$$\sup_{\mathbf{t} \in \mathcal{G}} \{g(\mathbf{t})\} = 0; \qquad (4)$$

$$g(\mathbf{0}) = 0 \qquad (5)$$

To introduce the notion of scale, we can make the above morphological operations scale dependent by defining a scaled structuring function $g_\sigma : \mathcal{G}_\sigma \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ such that [3]

$$g_\sigma(\mathbf{x}) = |\sigma| g(|\sigma|^{-1}\mathbf{x}) \quad \mathbf{x} \in \mathcal{G}_\sigma, \ \forall \sigma \neq 0, \qquad (6)$$

where $\mathcal{G}_\sigma = \{\mathbf{x} : \|\mathbf{x}\| < \mathcal{R}\}$ is the support region of the function g_σ [3].

To ensure reasonable scaling behavior, the following conditions are also necessary [3]:

$$|\sigma| \rightarrow 0 \Rightarrow g_\sigma(\mathbf{x}) \rightarrow \begin{cases} 0, & \text{if } \mathbf{x} = 0; \\ -\infty, & \text{otherwise} \end{cases} \qquad (7)$$

$$0 < |\sigma_1| < |\sigma_2| \Rightarrow g_{\sigma_1}(\mathbf{x}) \leq g_{\sigma_2}(\mathbf{x}) \text{ for } \mathbf{x} \in \mathcal{G}_{\sigma_1} \qquad (8)$$

$$|\sigma| \rightarrow \infty \Rightarrow g_\sigma(\mathbf{x}) \rightarrow 0 \quad \forall \quad \mathbf{x} \qquad (9)$$

The above conditions imply a monotonic decreasing structuring function along any radial direction from the origin [3].

Is this paper, we use as structuring function $g(x, y) = -\max\{x^2, y^2\}$. To make it scale dependent, we consider Equation 6 which yields:

$$g_\sigma(x, y) = -\frac{1}{|\sigma|} \max\{x^2, y^2\}, \qquad (10)$$

where σ represents the scale of the structuring function. Observe that, for a 3×3 structuring element, g_σ is zero at position $(0, 0)$ and $-\frac{1}{|\sigma|}$ otherwise. Figure 1 illustrates the structuring function.

In [3], Jackway uses the above concepts to define a transformation that smooths an image without introducing new extrema across different scales. This paper explores recursive applications of the above structuring function, having scale-space extrema preservation properties, to define a toggle transformation, as explained next.

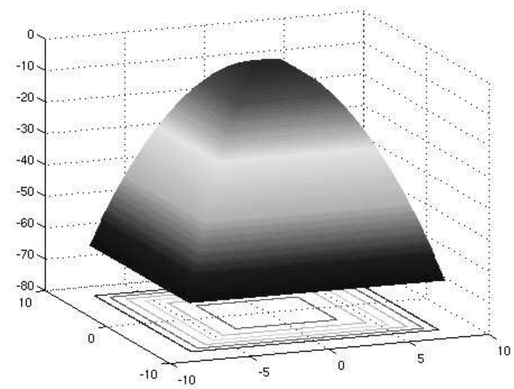


Figure 1. The structuring function.

3. Toggle Mapping

The key idea of toggle transformations or mappings is to associate an image with [20, 21]:

- a series of possible transformations ψ_i;
- a decision rule that determines at each pixel **x** the best value among the candidates $\psi_i(\mathbf{x})$.

The decision rule depends on the application. For a binary thresholding operation, for example, the decision rule involves, at point **x**, the value $f(\mathbf{x})$ and the threshold level. In this case, the primitives are the white and black images.

For contrast mapping, the primitives consist on one extensive and one anti-extensive transformation. The decision rule chooses the primitive value that is closest to the original image value. Toggle contrast operators based on erosion and dilation sharpen edges without boosting the contrast of image structures smaller than the structuring function being used [21, 17].

The primitives of a toggle operator can be independent of the initial image (as in thresholdings) or be themselves transformations acting on this initial signal (as in the morphological centers). Definition 3.1 presents a more formal definition discussed in [20].

Definition 3.1 *Let $\mathcal{F}(E;\bar{\mathbb{R}})$ be the class of the functions $f : E \Rightarrow \bar{\mathbb{R}}$, and $\mathcal{F}''$ be that of the mappings from $\mathcal{F}(E;\bar{\mathbb{R}})$ onto itself. Given a family (ψ_i) of elements of $\mathcal{F}''$, one calls toggle mapping of primitives (ψ_i) any mapping ω of $\mathcal{F}''$ such that:*

1. *at each point **x**, ω_x equals one of the $\psi_{i,\boldsymbol{x}}$ or $I_{\boldsymbol{x}}$,*
2. *the criterion which affects one of the ψ_i's, say ψ_{i_0} to ω at a given point **x** depends only on the various primitives ψ_i, on the numerical value $I_{\boldsymbol{x}}$ and on possible constants,*
3. *if at point **x**, at least one of the ψ_i's, say ψ_{i_0}, coincides with the identity mapping I, then*

$$\omega_{\boldsymbol{x}} = I_{\boldsymbol{x}} = \psi_{i_0,\boldsymbol{x}} \tag{11}$$

One way to guarantee a well controlled behavior of the whole process, with no risks of undesired spurious effects such as halos and/or oscillations, is to deal with idempotent toggle operators [20]. The next section illustrates another alternative to this problem based on a specific knowledge of the pixels transformation.

4. The Operator Definition

As stated before, a toggle operator has two major points: the primitives and a given decision rule. In this paper, we consider two primitives representing, respectively, an extensive and an anti-extensive transformation, namely, the scale dependent dilation and erosion (Equations 2 and 3). This toggle operator is given by

$$(f \oslash g_\sigma)^k(\mathbf{x}) = \begin{cases} \psi_1^k(\mathbf{x}) & \text{if } \psi_1^k(\mathbf{x}) - f(\mathbf{x}) < f(\mathbf{x}) - \psi_2^k(\mathbf{x}) \\ f(\mathbf{x}) & \text{if } \psi_1^k(\mathbf{x}) - f(\mathbf{x}) = f(\mathbf{x}) - \psi_2^k(\mathbf{x}) \\ \psi_2^k(\mathbf{x}) & \text{otherwise,} \end{cases} \tag{12}$$

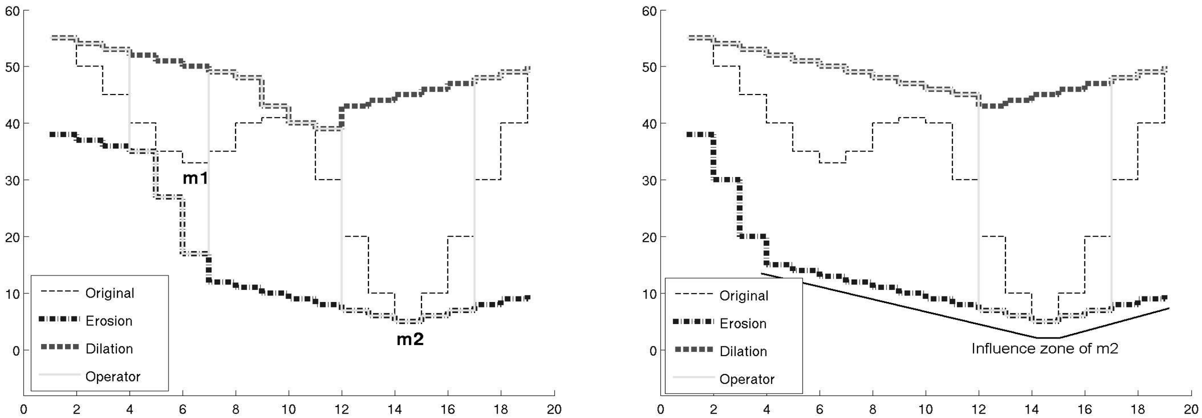
where $\psi_1^k(\mathbf{x}) = (f \oplus g_\sigma)^k(\mathbf{x})$, that is, the dilation of $f(\mathbf{x})$ with the scaled structuring function g_σ k times. In the same way, $\psi_2^k(\mathbf{x}) = (f \ominus g_\sigma)^k(\mathbf{x})$.

In a general way, the use of multiscale operators enables us to analyze the different representation levels to further choose the one exhibiting specific interest features. In our case, we explore the recursive applications of the primitives ψ_i^k, associated to the notion of scale, in order to decide the new state of a pixel.

Note that the above defined toggle operator is not idempotent. However, as explained in the next proposition, at a given point **x**, the operator will strictly increase (decrease) up to an iteration k_0, yielding a well-controlled toggling process. After this iteration, the value of **x** strictly decreases (increases).

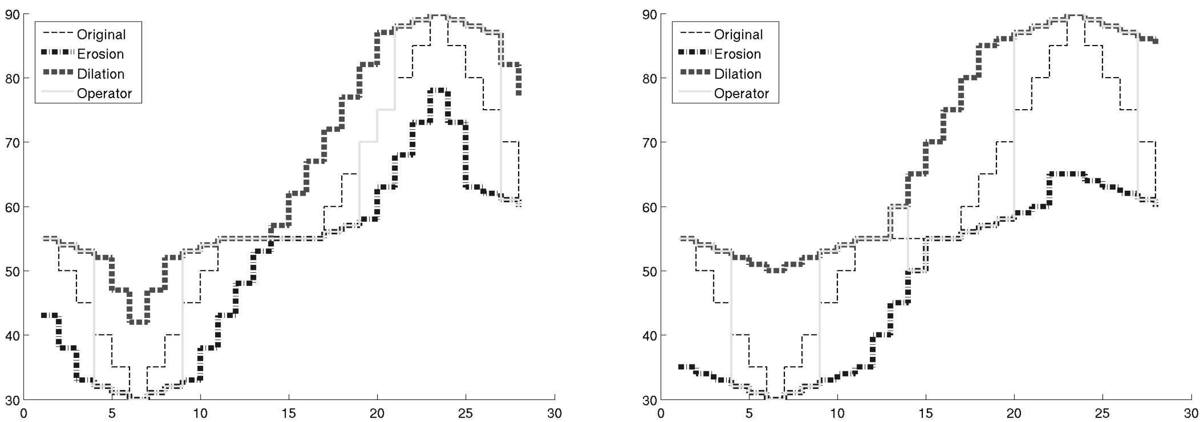
Proposition 4.1 *Let **x** be a given pixel in the image and $g(.)$ be as before a structuring function with a single maximum at the origin, that is, $g(\boldsymbol{x})$ is a local maximum implies $\boldsymbol{x} = \boldsymbol{0}$. The sequence defined by $(f \oslash g_\sigma)^k(\boldsymbol{x})$ is stationary and monotonic increasing (decreasing) until a certain iteration k_0, while it is monotonic decreasing (increasing) after the iteration k_0. For $k_0 = 1$ the sequence is strictly increasing (for dilations) and decreasing (for erosions). The proof of this proposition is given in Appendix A.*

Since the sequence is stationary, we have the guarantee that it converges to a constant value, that is, it stabilizes after a certain number of iterations.



(a) After seven iterations (b) After ten iterations

Figure 2. Operator behavior through different iterations using $g = [-1\ 0\ -1]$



(a) After three iterations (b) After five iterations

Figure 3. Operator behavior through different iterations using $g = [-1\ 0\ -1]$

Across the different iterations, a pixel can initially converge to a specific local minimum (maximum) and after a certain iteration, k_0, converges to an image maximum (minimum). In Figure 2, as the number of iterations increases, the influence zone of the deeper minimum m_2 growths and the eroded values in the neighborhood of m_1 decrease significantly, leading the value $f(m_1)$ to be closer to the dilated values.

Figure 3 shows an example in which $k_0 = 1$, that is, a pixel transformed value is strictly increasing or decreasing.

At this step, we can conclude that, in some neighborhood of an important minimum (maximum), the pixels values will be eroded (dilated) in such a way that, when these transformations are associated to the notion of scale, we can identify the significant extrema of the image and their influence zones through a simple thresholding operation.

In this sense, we define a new thresholding operation that uses the same primitives and decision rule as in Equation 12, given by:

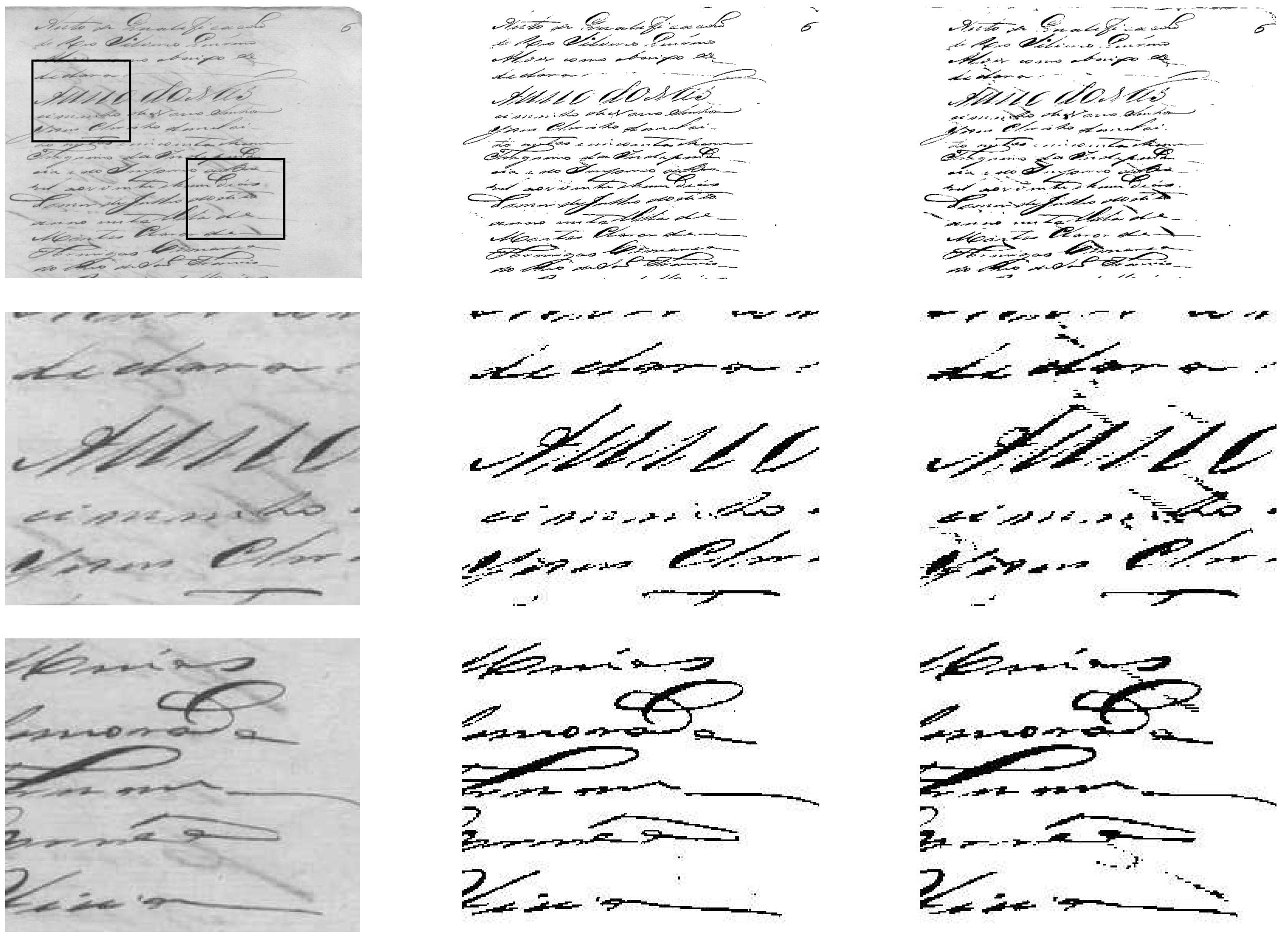
$$(f \oslash g_\sigma)^k(\mathbf{x}) = \begin{cases} 255 & \text{if } \psi_1^k(\mathbf{x}) - f(\mathbf{x}) <= f(\mathbf{x}) - \psi_2^k(\mathbf{x}) \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where, again, $\psi_1^k(\mathbf{x}) = (f \oplus g_\sigma)^k(\mathbf{x})$, that is, the dilation of $f(\mathbf{x})$ with the scaled structuring function g_σ k times. In the same way, $\psi_2^k(\mathbf{x}) = (f \ominus g_\sigma)^k(\mathbf{x})$.

5. Results

In this section, we give some examples of the transformation represented by Equation 13.

The first example shows the segmentation of a historical document in which the front side of the paper contains ink components from its verso side. The segmentation here was compared against the moving averages [22] algorithm, specially designed for segment text images. The moving



(a) Original image (b) Proposed operator (for $k = 5$ and $\sigma = 0.3$) (c) Moving average algorithm

Figure 4. Segmentation example for a historical document image

average method considers the mean gray level of the last n pixels. The pixels with a gray level lower than a fixed percentage of its moving average are set to black; otherwise they are set to white. Figure 4 shows the original and resulting images, and two selected regions. Note the better performance of our operator in the sense that it suppresses properly the components belonging to the reverse side of the paper.

The second experiment was carried out based on a set of images with varying lighting conditions (linear, Gaussian and sine-wave) [14], and on a set of well-known threshold-based segmentation methods described in literature, namely, moving averages [22], regional thresholds [14, 16], the Otsu's [12] and the Kapur's [5] thresholding algorithms. An evaluation of these methods as well as the set of considered images can be found in [14].

Briefly, Otsu's and Kapur's algorithms are based on the analysis of gray-level histograms. Otsu's method selects as an optimal threshold the one which minimizes the ratio between the "between-class" and the total variance. The between-class variance is defined as the deviation of the mean values for each considered class (background and object) from the overall mean of the pixels.

Kapur's algorithm takes into account the entropy of the gray level histograms. The method computes separately the entropy of object and background pixels, $H_b(t)$ and $H_w(t)$, choosing as the optimal threshold the value of t that maximizes $H = H_b(t) + H_w(t)$.

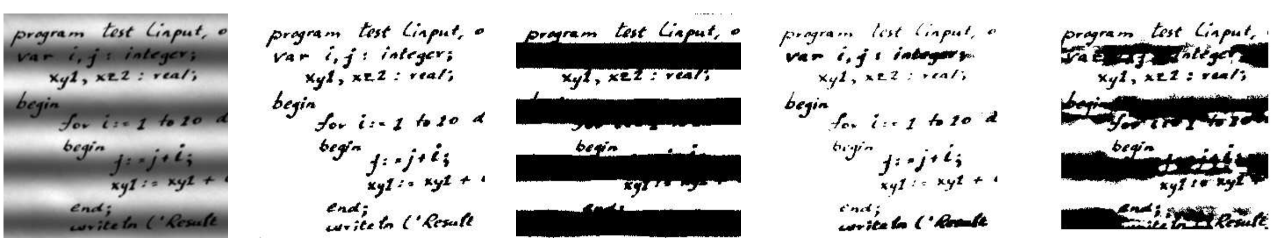
The approach based on regional thresholds divides the original image into regions and uses an appropriate algorithm to select a threshold per region. In this case, the size and the number of regions are important parameters, since we need to ensure that either region contains a sample of both object and background pixels. The examples illustrated in this paper were thresholded using the iterative selection [15] algorithm on overlapping 21×21 regions cen-

(a) Original image (b) Moving average (c) Otsu's method (d) Kapur's method (e) Regional thresholding

Figure 5. Segmentation results for the *face* image with linear illumination.

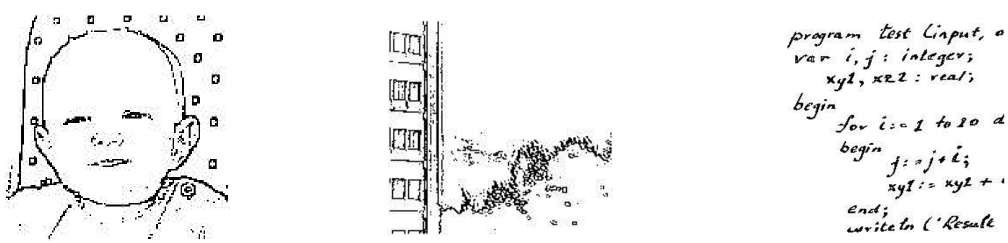
(a) Original image (b) Moving average (c) Otsu's method (d) Kapur's method (e) Regional thresholding

Figure 6. Segmentation results for the *sky* image with Gaussian illumination.



(a) Original image (b) Moving average (c) Otsu's method (d) Kapur's method (e) Regional thresholding

Figure 7. Segmentation results for the *pascal* image with sine-wave illumination.



(a) Face (for $k = 1$ and $\sigma = 0.1$) (b) Sky (for $k = 1$ and $\sigma = 0.06$) (c) Pascal (for $k = 2$ and $\sigma = 0.1$)

Figure 8. Segmentation results by the proposed operator for the original images in Figures 5-7.

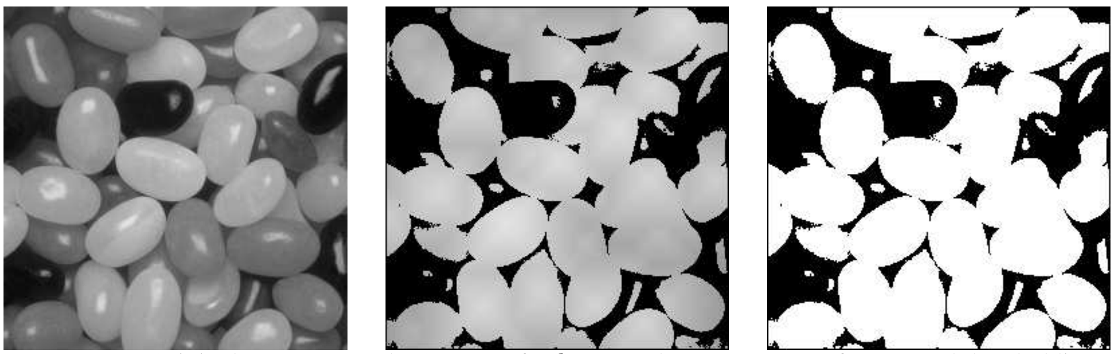
tered on each pixel. The iterative selection thresholds the image into object and background pixels repeatedly, using the levels in each class to refine the corresponding threshold.

Figures 5, 6 and 7 show the segmentation results for the above mentioned algorithms. Figure 8 presents the corresponding results using the operator defined by Equation 13. Note that the image background have converged to a local maximum (being assigned to 255).

The well-controlled behavior of the operator in Equation 13 is due to the merging of the image extrema so that no new maxima or minima are created, according to the morphological scale-space theory which constitutes the basis of our approach. The good performance of this approach can be explained mainly by the local information obtained from this well-controlled image extrema simplification, along the different iterations. Finally, note that the operator selects one threshold per pixel based on local features, as it is the case for dynamic thresholdings. Figures 9 shows other results for the operator in Equation 13.

(a) Lenna (b) Lenna for $k = 5$ and $\sigma = 2$ (c) Brain (d) Brain for $k = 50$ and $\sigma = 1$

Figure 9. Other examples of the operator in Equation 13.



(a) Original (b) Operator for $k = 5$ and $\sigma = 2$ (c) After manual thresholding

Figure 10. An example of the original image simplification (Equation 12).

6. Conclusions

In this work, we introduced a new toggle operator by exploring the strong monotonicity property for regions of a 2D signal, according to the morphological scale-space approach discussed in literature [3, 23]. We work with an explicit notion of scale guided by the scale-space theory, using a toggle transformation for segmentation problems, unlike the other applications of this operator which consider mainly problems related to image contrast enhancement.

This operator has a well-controlled behavior given by a merging of the image extrema in such a way that no new minima and maxima are created along its different iterations. Figure 10 illustrates this simplification process of an image, by progressive merging of extrema, with preservation of its main geometric features. Further, by taking into consideration some aspects of this merging, we proposed a binarization procedure (Equation 13), yet in the scope of a toggle mapping, to achieve the final segmentation.

As a future work, we will deal with the problem of locally controlling the extrema merging by taking into account the height of the structuring functions and the distance between the image extrema in the definition of a parametric mapping using both these information.

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Appendix A: Proof of proposition

Proposition 1

The operator defined in Equation 12 can be rewritten as

$$(f \oslash g_\sigma)(\mathbf{x}) = \begin{cases} \psi_1^k(\mathbf{x}) & \text{if } f(\mathbf{x}) > \frac{1}{2}(\beta_k + \alpha_k) \\ f(\mathbf{x}) & \text{if } f(\mathbf{x}) = \frac{1}{2}(\beta_k + \alpha_k) \\ \psi_2^k(\mathbf{x}) & \text{otherwise} \end{cases} \quad (14)$$

with $\alpha_k = \max_{\mathbf{t} \in N(\mathbf{x},j), \mathbf{t}_j \neq 0} \{f(\mathbf{x}), f(\mathbf{x} - \mathbf{t}_j) + jg(\mathbf{t})\}$ and $\beta_k = \min_{\mathbf{t} \in N(\mathbf{x},j), \mathbf{t}_j \neq 0} \{f(\mathbf{x}), f(\mathbf{x} - \mathbf{t}_j) - jg(\mathbf{t})\}$ where $j = 1, 2, \ldots, k$, with k the number of iterations, $N(\mathbf{x}, \epsilon)$ are the set of pixels that are in a chess distance less or equal ϵ from $\mathbf{x}$.

The sequence (α_k) satisfies:

$\alpha_k \geq f(\mathbf{x})$;

(α_k) is monotonically increasing, that is, $\alpha_k \leq \alpha_{k+1} \ \ \forall k$;

(α_k) is stationary, that is, $\exists\ k_0/\forall\ k \geq k_0,\ \alpha_k = \alpha_{k_0}$.

The sequence (β_k) satisfies:

$\beta_k \leq f(\mathbf{x})$;

(β_k) is monotonically increasing, that is, $\beta_k \geq \beta_{k+1} \ \ \forall k$;

(β_k) is stationary, that is, $\exists\ k_0/\forall\ k \geq k_0,\ \beta_k = \beta_{k_0}$.

Let $\gamma_k = \frac{1}{2}(\alpha_k + \beta_k)$. As the sequences (α_k) and (β_k) are stationary and monotone, we have that the sequence (γ_k) is both monotonic and stationary [10]. This yields a sequence γ_k that is monotonically increasing or decreasing and that stabilizes after a certain number of iterations.

It worths noting that, depending on the (α_k) and (β_k) values, the sequence (γ_k) can intercept the value $f(x_0)$ across iterations. Figure 11 shows an example in which the operator result is given by dilation, until iteration k_0, and then by erosion after this iteration. When this happens, we know that the pixel is being affected by an influence zone of a stronger image extremum (it changes from a maximum to a minimum influence zone).

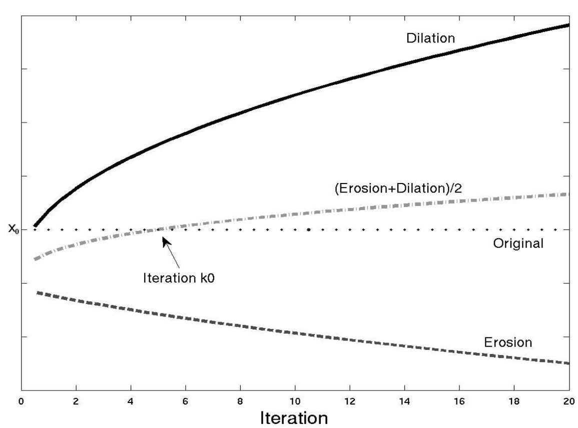

Figure 11. Proposition 1 illustration.

References

[1] R. Haralick, S. Stenberg, and X. Zhuang. Image analysis using mathematical morphology. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-9(4):532–550, 1987.

[2] P. Jackway. Gradient watershed in morphological scale-space. *IEEE Transactions on Image Processing*, 15:913–921, 1996.

[3] P. Jackway and M. Deriche. Scale-space proprieties of the multiscale morphological dilation-erosion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 18:38–51, 1996.

[4] B. Jang and R. Chin. Morphological scale-space for 2d shape smoothing. *Computer Vision and Image Understanding*, 70(2):121–141, 1998.

[5] J. Kapur, P. Sahoo, and A. Wong. A new method for gray-level picture thresholding using the entropy of the histogram. *Computer Vision, Graphics, and Image Processing*, 29:273–285, 1985.

[6] H. P. Kramer and J. B. Bruckner. Iterations of a non-linear transformation for enhancement of digital images. *Pattern Recognition*, 7:53–58, 1975.

[7] N. J. Leite and M. D. Teixeira. Morphological scale-space theory for segmentation problems. In *IEEE-EURASIP Workshop on Nonlinear Signal and Image Processing*, pages 364–368, 1999.

[8] N. J. Leite and M. D. Teixeira. An idempotent scale-space approach for morphological segmentation. In *Mathematical Morphology and its Applications to Image and Signal Processing*, pages 291–300. Kluwer Academic Publishers, 2000.

[9] L. Lifshitz and S. Pizer. A multiresolution hierarchical approach to image segmentation based on intensity extrema. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12(4):529–540, 1990.

[10] J. Marsden and M. Hoffman. *Elementary Classical Analysis*. Freeman, 1993.

[11] G. Matheron. *Random Sets and Integral Geometry*. John Wiley and Sons, 1975.

[12] N. Otsu. A threshold selection method from grey-level histograms. *IEEE Transactions on Systems, Man and Cybernetics*, 9(1):377–393, 1979.

[13] K. Park and C. Lee. Scale-space using mathematical morphology. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 18(11):1121–1126, 1996.

[14] J. Parker. *Algorithms for Image Processing and Computer Vision*. Wiley, 1996.

[15] T. Ridler and S. Calvard. Picture thresholding using a iterative selection method. *IEEE Transactions on Systems, Man and Cybernetics*, SMC-8(8):233–260, 1978.

[16] A. Rosenfeld and A. Kak. *Digital Picture Processing*. Academic Press, 1982.

[17] J. G. M. Schavemaker, M. J. T. Reinders, J. Gerbrands, and E. Backer. Image sharpening by morphological filtering. *Pattern Recognition*, 33:997–1012, 1999.

[18] J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press, 1982.

[19] J. Serra. *Image Analysis and Mathematical Morphology, volume 2: Theoretical Advances.* Academic Press, 1988.

[20] J. Serra and L. Vicent. An overview of morphological filtering. *Circuits, Systems and Signal Processing*, 11(1):47–108, 1992.

[21] P. Soille. *Morphological Image Analysis: Principles and Applications*. Springer-Verlag, 2003.

[22] P. Wellner. Adaptive thresholding for the digital desk. Technical Report EPC1993-110, Xerox, 1993.

[23] A. P. Witkin. Scale-space filtering: a new approach to multiscale description. In *Image Understanding*, pages 79–95. Ablex, 1984.